CONTROL OF A SIMPLE CASE OF INHERENTLY UNSTABLE SYSTEMS

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ENGINEERING ELECTRONICS

United States Naval Postgraduate School
Monterey, California

1965

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ABSTRACT

Here is a simple example of control of inherently unstable system. An inverted pendulum pivoted on top of a cart is to be stabilized by applying force to the cart through an electric motor.

In the electrical laboratory of the United States Naval Postgraduate School, a cart with a stick pivoted on top of it has been built, tested and simulated with CDC 1604 digital computer.

The author, Lieutenant Mu-yu Wan of the Chinese Navy, wishes to thank Dr. Harold A. Titus of the United States Naval Postgraduate School for his patient assistance in this work as thesis supervisor.

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1. Introduction

It seems to be interesting to provide artificial stability for an inherently unstable physical system. Some immediate questions arise, such as: what are the largest errors which can possibly be corrected with a limited available control force, and what is the best control strategy which minimizes the time required and maximizes the size of the system errors which can be corrected.

In this simple case here, an inverted pendulum pivoted on top of a cart is to be stabilized by applying limited force to the cart through an electric motor. This inherently unstable system is assumed to be adequately represented by a set of linear differential equations over the range of interest. The type of instability is represented by real non-negative characteristic roots. The motor supply voltage is manipulated, within its allowable limits, as a function of the state variables of the set of linear differential equations.

In Chapter 3, an analog computer is used to really balance the pendulum. The general schematic of the system is shown here.

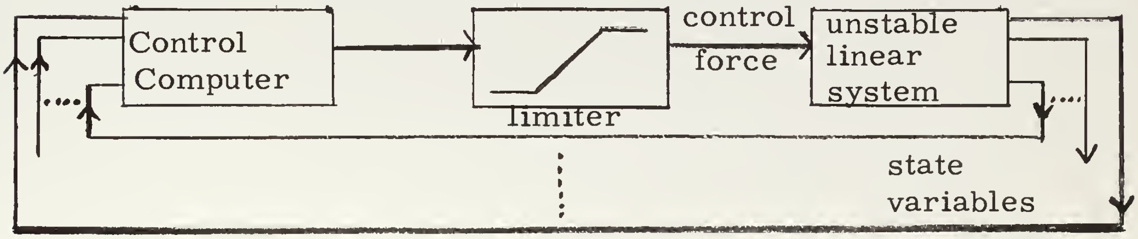


FIG. (1-1). General Schematic

In Chapter 4, the whole system is simulated with CDC 1604 digital computer, Graphs are plotted and situations discussed.

In Chapter 5, the technique of optimal discrete-time control is introduced. The results simulated by CDC 1604 turned out to be successful.

2. Uncontrolled System

2. 1 Linear Differential Equations

As shown in Fig.(2-1-1), the instantaneous angle that the pendulum makes with its unstable equilibrium position is θ, and the position of the cart with respect to some reference point on the floor is ξ.. The coordinates are shown with positive displacement.

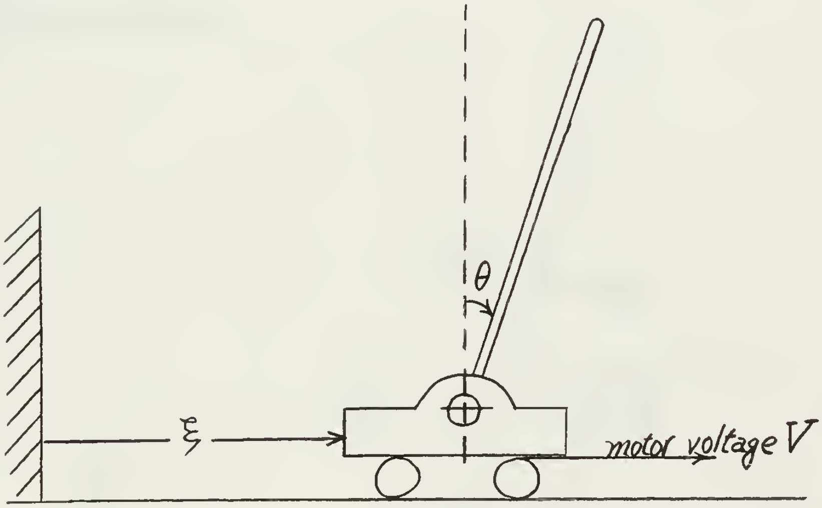


FIG. (2-1-1). System To Be Stabilized

For establishing the equations of motion, we define some additional symbols:

f	force applied to cart
$f_{\dot{\xi}}$	damping coefficient including friction and back e. m. f.
fv	applied voltage force coefficient

g	acceleration of gravity
M	total system effective mass
m	mass of the pendulum
r	distance of pendulum mass center from hinge line
ρ	pendulum radius of gyration about hinge line
V	voltage applied to d. c. drive motor

The equations of motion can be found by consideration of the following diagram.

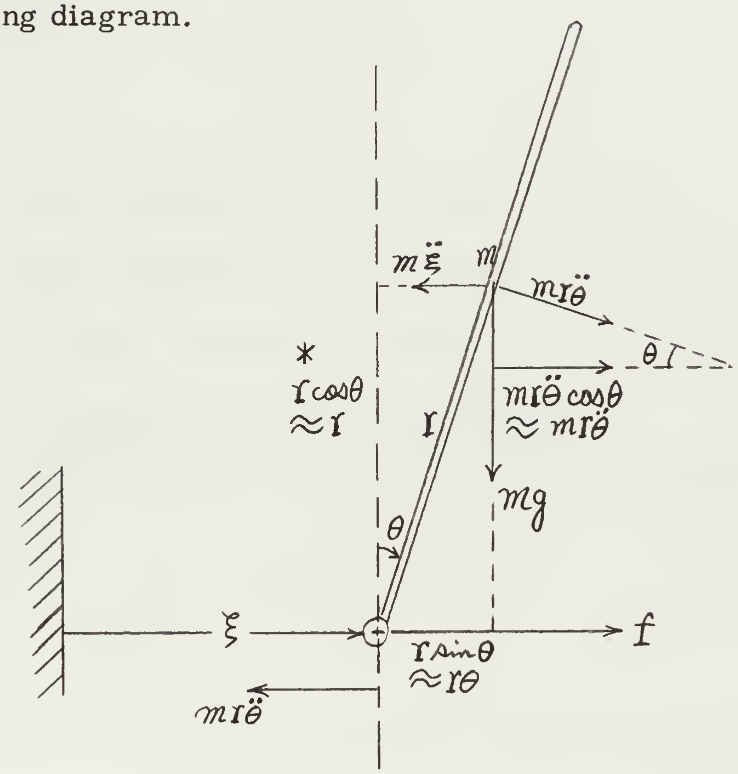


FIG. (2-1-2). Force Diagram Showing Equations of Motion
* Note: θ is very small.

The liniarized equations of motion are

$$m\rho^2\ddot{\theta} = mrg\theta - mr\ddot{\xi} \qquad (2.1.1)$$

and

$$M\ddot{\xi} = -mr\ddot{\theta} + f \tag{2.1.2}$$

where

$$f = f_{\dot{\xi}}\dot{\xi} + f_V V \tag{2.1.3}$$

We have data measurements as follows:

$g = 9.8\ ^{m}/_{sec^2}$

$m = 0.225$ kg.

$M = 6.0$ kg.

$r = 0.44$ m.

l = length of longer part from hinge = 0.94 m.

h = length of shorter part from hinge = 0.035 m.

$\rho^2 = (l^2 + 3h^2 - 3lh) \times {}^1/_3 = 0.25\ m^2$

V = 24 volts (for selected D. C. motor)

The force exerted on the cart is 13.6 newton while the cart is held still, i.e., $\dot{\xi} = 0$, thus from (2.1.3)

$$f_V = \frac{f}{V} = \frac{13.6\ n.}{24\ v} = 0.57\ ^{n}/_{v}$$

In (2.1.2), we have mass of the pendulum much less than mass of the whole system, the term "$mr\ddot{\theta}$" can be neglected, given:

$$M\ddot{\xi} = f = f_{\dot{\xi}}\dot{\xi} + f_V V \tag{2.1.4}$$

Securing the pendulum ($\ddot{\theta} = 0$), we run the cart on the floor and observe the motion with a brush recorder, find the average velocity and acceleration as:

$$\dot{\xi} = 0.90 \ {}^{m}/{}_{sec}$$

and

$$\ddot{\xi} = 0.83 \ {}^{m}/{}_{sec}$$

By (2. 1. 4)

$$f_{\dot{\xi}} = \frac{M\ddot{\xi} - f_v V}{\dot{\xi}} = -11.8 \ {}^{n\text{-}sec}/{}_{m.}$$

Now (2. 1. 1) and (2. 1. 4) can be written as:

$$\ddot{\theta} = \frac{rg}{\rho^2}\theta - \frac{r}{\rho^2}\ddot{\xi} \tag{2.1.1}$$

$$\ddot{\xi} = \frac{f_{\dot{\xi}}}{M}\dot{\xi} + \frac{f_V}{M}V \tag{2.1.4}$$

Define:

$$\left.\begin{aligned} \theta &= \theta_1 \\ \dot{\theta} &= \dot{\theta}_1 = \theta_2 \\ \xi &= \xi_1 \\ \dot{\xi} &= \dot{\xi}_1 = \xi_2 \end{aligned}\right\}$$

Then, we get a set of linear differential equations as:

$$\left.\begin{aligned} \dot{\theta}_1 &= \theta_2 \\ \dot{\theta}_2 &= \frac{rg}{\rho^2}\theta_1 - \frac{r}{M\rho^2}\left[f_{\dot{\xi}_1}\xi_2 + f_V V\right] \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{1}{M}\left[f_{\dot{\xi}_1}\xi_2 + f_V V\right] \end{aligned}\right\}$$

In matrix form:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{rg}{\rho^2} & 0 & 0 & -\frac{rf_{\dot{\xi}}}{M\rho^2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{f_{\dot{\xi}}}{M} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{rg}{\rho^2} \\ 0 \\ g \end{bmatrix} \frac{Vf_V}{Mg} \tag{2.1.5}$$

Substituting numberical values:

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 17.2 & 0 & 0 & 3.5\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -2\end{bmatrix} \times \begin{bmatrix}\theta_1\\ \theta_2\\ \xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}0\\ -17.2\\ 0\\ 9.8\end{bmatrix} \frac{V f_v}{Mg} \qquad (2.1.5)$$

Or, by defining some new matrix names, and x's as the name of state variables:

$$\dot{\underline{x}} = [A]\,\underline{x} + \underline{c}\,u \qquad (2.1.6)$$

where

$$u = \frac{V f_v}{Mg} \qquad (2.1.7)$$

$$[A] = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{rg}{\rho^2} & 0 & 0 & -\frac{r f_{\dot{\xi}}}{M\rho^2}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & \frac{f_{\dot{\xi}}}{M}\end{bmatrix} \qquad (2.1.8)$$

$$\dot{\underline{x}} = \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} \qquad \underline{x} = \begin{bmatrix}\theta_1\\ \theta_2\\ \xi_1\\ \xi_2\end{bmatrix} \qquad \underline{c} = \begin{bmatrix}0\\ -\frac{rg}{\rho^2}\\ 0\\ g\end{bmatrix}$$

2.2 Transformation to Canonical Form

Assuming in our linear transformation matrix $[A]$ there is some eigenvector $\underline{v}$ associated with eigenvalue λ:

$$[A]\,\underline{v} = \lambda\,\underline{v} \qquad \text{where } \underline{V} \neq 0$$

$$[A - \lambda I]\,\underline{v} = 0$$

or

$$|A - \lambda I| = 0$$

By (2.1.8)

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ \frac{rg}{\rho^2} & -\lambda & 0 & -\frac{r f_{\dot\xi}}{M\rho^2} \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & \frac{f_{\dot\xi}}{M} - \lambda \end{vmatrix} = 0$$

There comes the characteristic equation:

$$\lambda\left(\lambda - \frac{f_{\dot\xi}}{M}\right)\left(\lambda^2 - \frac{gr}{\rho^2}\right) = 0$$

The eigenvalues are:

$$\left.\begin{aligned} \lambda_1 &= -\sqrt{\frac{gr}{\rho^2}} = -4.15 \\ \lambda_2 &= +\sqrt{\frac{gr}{\rho^2}} = +4.15 \\ \lambda_3 &= 0 \\ \lambda_4 &= \frac{f_{\dot\xi}}{M} = -2 \end{aligned}\right\}$$

Only λ_2 and λ_3 are non-negative or unstable.

With these eigenvalues all distinct, one can always find a new set of state variables:

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

related to $\underline{x}$ by the transformation

$$\underline{y} = [G]\,\underline{x} \tag{2.2.1}$$

such that the system of equations (2.1.6) transforms to:

$$\dot{\underline{y}} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{bmatrix} \underline{y} + \underline{D}\,u \tag{2.2.2}$$

with the 4 x 4 matrix $[G]$ and vector $\underline{D}$ given later.

From (2.2.1)

$$\underline{x} = [G]^{-1}\,\underline{y} \tag{2.2.3}$$

Let

$$[G]^{-1} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

The transformation matrix $[G]^{-1}$ is not unique, but a convenient form for this problem can be found by four column vectors, all are eigenvectors associated with one eigenvalue respectively.

Define

$$\left[G\right]^{-1} \stackrel{D}{=} \left[\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3 \quad \underline{v}_4\right] \tag{2.2.4}$$

Where

$$\left[A - \lambda_i \, I\right] \underline{v}_i = 0 \qquad\qquad i = 1,\ 2,\ 3,\ 4 \tag{2.2.5}$$

But $\lambda_1 = -\lambda_2 = -\sqrt{\frac{gr}{\rho^2}}$, $\lambda_4 = \frac{f_\xi}{M}$, (2.1.8) appears as:

$$\left[A\right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_1^2 & 0 & 0 & -\frac{\lambda_4 \lambda_1^2}{g} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{2.1.8}$$

Solving (2.2.5) for i = 1

$$\begin{bmatrix} -\lambda_1 & 1 & 0 & 0 \\ \lambda_1^2 & -\lambda_1 & 0 & -\frac{\lambda_4 \lambda_1^2}{g} \\ 0 & 0 & -\lambda_1 & 1 \\ 0 & 0 & 0 & -\lambda_1 + \lambda_4 \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{41} \end{bmatrix} = 0$$

There are many possible solutions, one set of which can be:

$$\left.\begin{aligned} g_{41} &= 0 \\ g_{31} &= 0 \\ g_{21} &= -\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \\ g_{11} &= -\frac{\lambda_2}{\lambda_2 - \lambda_1} \end{aligned}\right\}$$

for i = 2, 3, 4 respectively (note $\lambda_3 = 0$), we can adopt:

$$\left.\begin{aligned} g_{42} &= 0 \\ g_{32} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} g_{22} &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \\ g_{12} &= \frac{\lambda_1}{\lambda_2 - \lambda_1} \end{aligned}\right\}$$

$$\left.\begin{aligned} g_{43} &= 0 \\ g_{33} &= -g/\lambda_4 \\ g_{23} &= 0 \\ g_{13} &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} g_{44} &= g/\lambda_4 \\ g_{34} &= g/\lambda_4^2 \\ g_{24} &= \frac{-\lambda_1^2 \lambda_4}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)} \\ g_{14} &= \frac{-\lambda_1^2}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)} \end{aligned}\right\}$$

Then

$$[G]^{-1} = \begin{bmatrix} \frac{-\lambda_2}{\lambda_2 - \lambda_1} & \frac{\lambda_1}{\lambda_2 - \lambda_1} & 0 & \frac{-\lambda_1^2}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)} \\ \frac{-\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} & \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} & 0 & \frac{-\lambda_1^2 \lambda_4}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)} \\ 0 & 0 & -\frac{g}{\lambda_4} & \frac{g}{\lambda_4^2} \\ 0 & 0 & 0 & \frac{g}{\lambda_4} \end{bmatrix}$$

Inverse of $[G]^{-1}$ is $[G]$:

$$[G] = \begin{bmatrix} -1 & -\frac{1}{\lambda_1} & 0 & \frac{1}{g}\,\frac{\lambda_1 \lambda_4}{\lambda_1 - \lambda_4} \\ -1 & -\frac{1}{\lambda_2} & 0 & \frac{1}{g}\,\frac{\lambda_2 \lambda_4}{\lambda_2 - \lambda_4} \\ 0 & 0 & -\frac{\lambda_4}{g} & \frac{1}{g} \\ 0 & 0 & 0 & \frac{\lambda_4}{g} \end{bmatrix} \qquad (2.\,2.\,6)$$

By (2.1.6)

$$\dot{\underline{x}} = [A]\,\underline{x} + \underline{c}\,u \tag{2.1.6}$$

By transformation

$$\underline{y} = [G]\,\underline{x}$$

$$[G]^{-1}\dot{\underline{y}} = [A][G]^{-1}\underline{y} + \underline{c}\,u$$

$$\dot{\underline{y}} = [G][A][G]^{-1}\underline{y} + [G]\,\underline{c}\,u \tag{2.2.7}$$

Define

$$[G][A][G]^{-1} \overset{D}{=} [J] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{bmatrix} \tag{2.2.8}$$

$$[G]\cdot\underline{c} \overset{D}{=} \underline{D} \overset{D}{=} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{1-\lambda_4/\lambda_1} \\ \frac{\lambda_2}{1-\lambda_4/\lambda_2} \\ 1 \\ \lambda_4 \end{bmatrix} \tag{2.2.9}$$

Finally, we get the Jordan Canonical form of the system:

$$\dot{\underline{y}} = [J]\,\underline{y} + \underline{D}\,u \tag{2.2.10}$$

2.3 The Reduced System of Equations

In the last section, the whole system is represented by equation (2.2.10). Now we have to show that for purpose of establishing a controller which assures stability of the equilibrium point at the origin, it is sufficient to consider the following reduced system.

$$\dot{y}_i = \lambda_i y_i + d_i u \qquad i = 2,3 \qquad (2.3.1)$$

without regard for the coordinates y_1 and y_4, which associated with negative eigenvalues.

If this is true, i.e., a controller $u = f(y_2, y_3)$ can be found which makes the system (2.3.1) asymptotically stable for some region about the origin of the two dimensional space. By definition:

$$y_i \to 0 \quad \text{as } t \longrightarrow \infty \qquad i = 2, 3$$

From (2.3.1)

$$(\dot{y}_i - d_i u) \to 0 \quad \text{as } t \to \infty \qquad i = 2, 3$$

Then, certainly, as $t \to \infty$

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} (\dot{y}_i - d_i u)\,dt \to 0 \qquad \Delta t \neq 0 \qquad i = 2, 3$$

But

$$\lim_{t\to\infty} \frac{1}{\Delta t}\int_t^{t+\Delta t} (\dot{y}_i - d_i u)\,dt = \lim_{t\to\infty}\left\{ \frac{y_i(t+\Delta t) - y_i(t)}{\Delta t} - d_i \frac{1}{\Delta t}\int_t^{t+\Delta t} u\,dt \right\}$$

Then

$$\lim_{t\to\infty} \int_t^{t+\Delta t} u\,dt \longrightarrow 0 \qquad \Delta t \neq 0$$

This implies $u(t) \rightarrow 0$ as $t \rightarrow \infty$ in the sense of its average value over any very small non-zero time interval.

Now, we come back to those equations for y_1 and y_4. By means of Jordan canonical form, the state variables are expressed by partial fraction as the following block diagram.

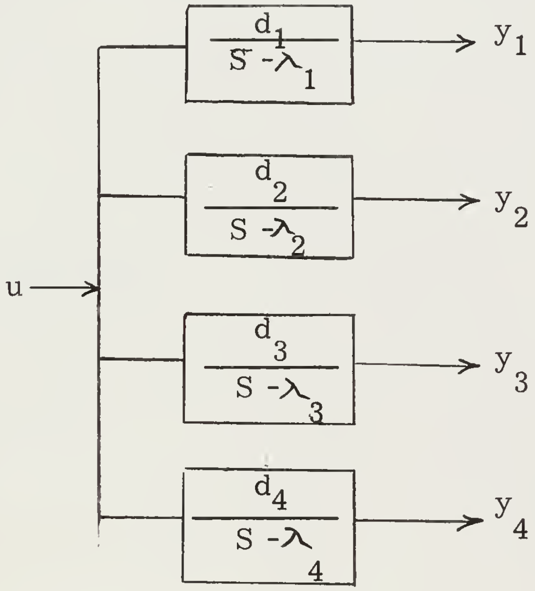


FIG. (2-3-1). State Variables in Jordan Canonical Form

Let $h_j(t)$ be the impulse response for y_j, then

$$\lim_{t\to\infty} y_j(t) = \lim_{t\to\infty}\int_0^t h_j(t-t_1)\, u(t_1)\, dt \qquad j = 1, 4$$

Since $h_j(t)$ approaches zero exponentially as $t \rightarrow \infty$ (because of negative eigenvalue), and since as mentioned above, $u(t)$ can be considered to approach zero under any integral sign, it follows that $y_j \rightarrow 0$ as $t \rightarrow \infty$.

Thus the initial displacement of the states y_1 and y_4 have no effect on the region of stability of the complete system. Therefore, from now on, we only consider the reduced system described by equation (2.3.1).

3. Control with an Analog Computer

3.1 Selection of a Controller

A controller is needed to provide stability in the region of controllability, which means the largest region in the state space within which the system can be brought to the point of equilibrium at the origin with the constraint $u \leqslant U$. The controller will be a function only of y_2 and y_3, but through the transformation $\underline{y} = [G]\underline{X}$, it will generally involve all the state variables of the original system. Pontryagin's maximum principle determines an optimum $u(t)$ which minimizes:

$$y_0(T) = \int_0^T f(y_2, y_3)\, dt$$

with the constraint

$$|u| \leqslant U$$

and final states

$$y_i(T) = 0 \qquad i = 2, 3$$

f_0 is some positive cost function, different kinds of which lead to different kinds of controller. For the case of minimum-time controller $f_0 = 1$. We define a new state variable:

$$y_4 = y_0(T) = \int_0^T dt$$

It follows

$$\dot{y}_4 = 1$$

By adding this new state variable, our system gets:

$$\begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} \lambda_2 y_2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} d_2 u \\ u \\ 0 \end{bmatrix} \qquad (3.1.1)$$

Pontryagin further defines the Hamiltonian function, maximizing this H function with respect to u has the same effect as minimizing $y_0(T)$.

In our case:

$$H=\sum_{i=2}^{4} P_i \dot{y}_i = P_2(\lambda_2 y_2 + d_2 u) + P_3 u + P_4$$

For maximization with respect to u, probably the bang-bang type controller ($u = \pm U$) is the best consideration.

Let

$$u = U \operatorname{sgn}\left[f^*\right]$$

Usually f^* can be derived by solving the following canonical equations.

$$\dot{y}_i = \frac{\partial H}{\partial P_i}$$

$$\dot{P}_i = -\frac{\partial H}{\partial y_i}$$

In this problem, it is hard to express them explicitly. In view of piecewise linear switching, the following guess seems reasonable.

$$f^* = b\, y_3 - y_2 \qquad (3.1.2)$$

where b is some positive constant.

Therefore

$$u = U \operatorname{sgn}\left[b y_3 - y_2\right] \qquad (3.1.3)$$

If there is no control on cart position, that means ξ is no longer a state variable. Then by equations (2. 2. 1) and (2. 2. 6) the uncoupled state y_3 has no meaning. In this case the whole system (2. 3. 1) reduces to:

$$\dot{y}_2 = \lambda_2 y_2 + d_2 u \qquad (3.1.4)$$

$$u = U \operatorname{sgn}\left[-y_2\right] \qquad (3.1.5)$$

3. 2 Realization of the Control

Now, we get a minimum-time controller, which is (through the functions of y_2 and y_3) in terms of the original state variables, namely, θ_1, θ_2, ξ_1 and ξ_2. Those state variables can be generated by two potentiometers and two techometers. We select a Donner Analog Computer to sum them up and use a D. C. relay to generate sgn function. The real structure is shown as Fig.(3-2-1).

3. 2. 1 No Control of Cart Position

By (3. 1. 5)

$$u = U \operatorname{sgn}\left(-y_2\right) \qquad (3.1.5)$$

By (2. 2. 6)

$$y_2 = -\theta_1 - \frac{1}{\lambda_2}\theta_2 + \frac{1}{g}\frac{\lambda_2\lambda_4}{\lambda_2-\lambda_4}\xi_2$$

$$= -\theta_1 - 0.241\,\theta_2 - 0.138\,\xi_2 \qquad (3.2.1.1)$$

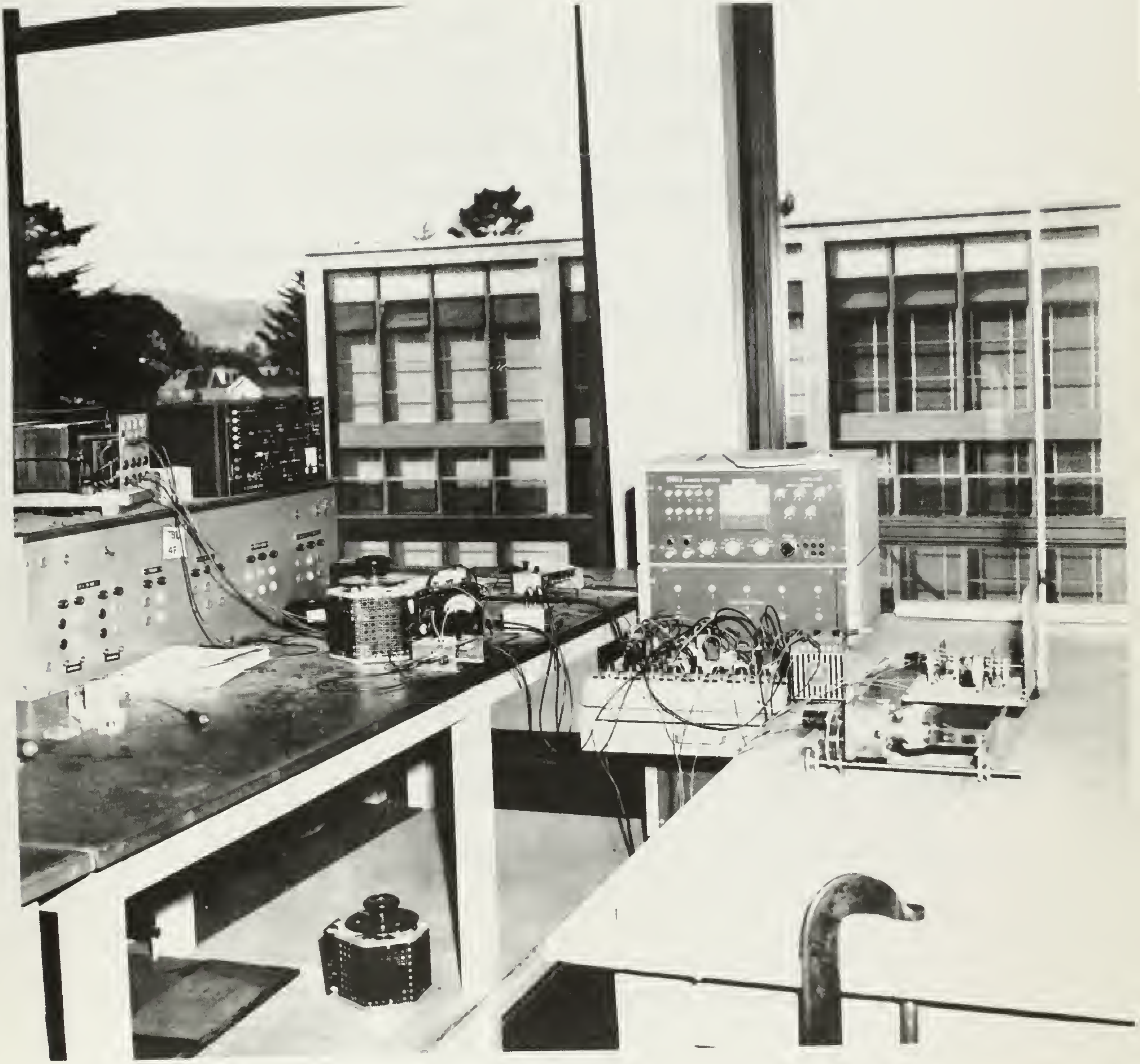


FIG. (3-2-1). Cart Being Controlled by Analog Computer

By measuring those potentiometer and tachometers, the following data are obtained.

θ: 1 volt corresponds to 0. 087 radian

$\dot{\theta}$: 1 volt corresponds to 40. 4 radian/sec

$\dot{\xi}$ 1 volt corresponds to 0. 36 meter/sec

Multiplying these factors, we get:

$$u = U \operatorname{sgn}\left[0.087\ \theta_1 + 9.75\ \theta_2 + 0.05\,\xi_2\right] \qquad (3.2.1.2)$$

where

$$U = \frac{V f_v}{M g} = 0.24 \qquad (3.2.1.3)$$

The circuitry is built for the controller as shown in Fig. (3-2-1-1).

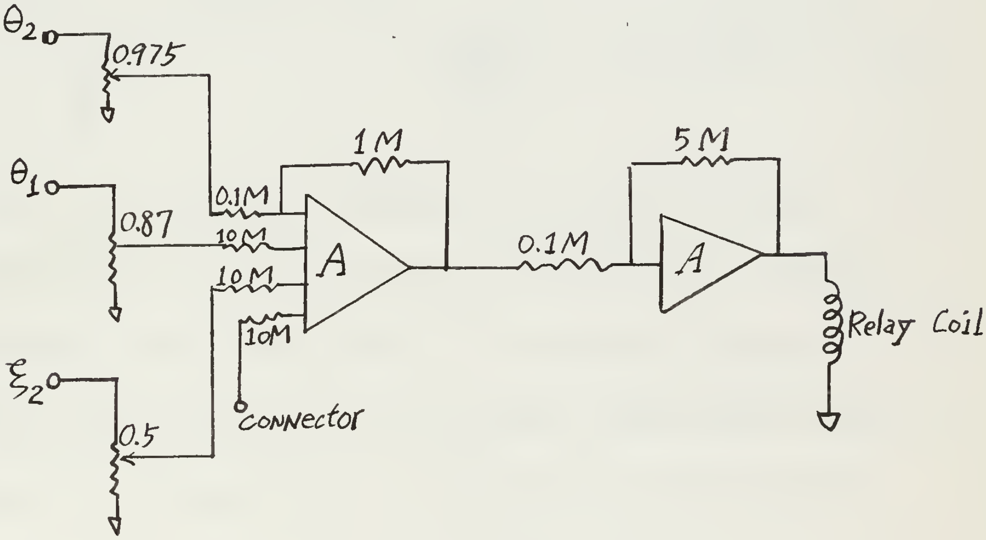


FIG. (3-2-1-1). Controller Circuitry

3. 2. 2 With Control of Cart Position

By (3. 1. 3)

$$u = U \operatorname{sgn}\left[b y_3 - y_2\right] \qquad (3.1.3)$$

From (2. 2. 6)

$$y_3 = -\frac{\lambda_4}{g}\xi_1 + \frac{1}{g}\xi_2 = 0.204\,\xi_1 + 0.102\,\xi_2 \qquad (3.2.2.1)$$

For ξ:

1 volt corresponds to 0. 05 meter

Then

$$y_3 = 0.0102\,\xi_1 + 0.036\,\xi_2$$

The following network is added to the connector in Fig. (3-2-1-1)

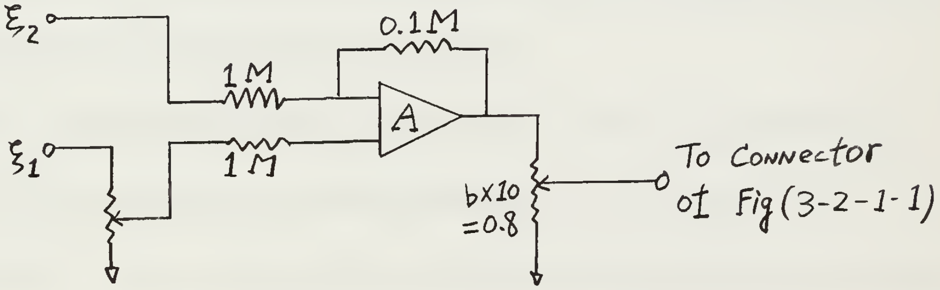


FIG. (3-2-2-1). Additional Network for Position Control

If the value of b is not big enough, the cart has a tendency to settle itself a little bit off-center. For instance, taking b=0. 08 (as shown in Fig. 3-2-2-1), the cart tends to settle itself about 0. 2 meter from its starting position. If the polarity of the potentiometer for ξ is reversed, the cart will tend to settle in the opposite side from the starting position. Anyhow, if b is big enough, the cart will come back to its starting position. This is shown in appendix II-5, for b = 0. 7.

4. Simulation by Graphs

4.1 The Region of Controllability

When we are playing with the physical cart, we can start the cart motion by just pushing the pendulum off its vertical position. Now, for the case of simulation with digital computer, we encounter the problem of deciding the initial condition of θ_1. In other words, we want to know the region of controllability which means the largest region in the state space from which the system still can be brought back to its equilibrium point.

With a bang-bang type controller ($u = \pm U$), the trajectories consist of two segments corresponding to $u = U$ and $u = -U$ respectively. The origin of the two-dimensional space can always be reached by this switching of u. The region of controllability, then, can be defined as the set of points reachable by the trajectory starting at the origin ($y_2 = 0$, $y_3 = 0$), and proceeding in reverse time ($0 \leqslant t \leqslant -\infty$) with u alternately taking values of +U and -U. In our problem:

$$\left.\begin{aligned} \dot{y}_2 &= \lambda_2 y_2 + d_2 u \\ \dot{y}_3 &= d_3 u \end{aligned}\right\} \qquad (2.3.1)$$

Those first order differential equations can be easily solved with solutions as:

$$\frac{y_2 + (d_2u/\lambda_2)}{y_{20} + (d_2u/\lambda_2)} = \exp\left[\lambda_2(t-t_o)\right] \qquad (4.1.1)$$

$$\frac{y_3 - y_{30}}{d_3u} = t - t_o \qquad (4.1.2)$$

From (4. 1. 2), y_3 is undefinted as $t \to -\infty$, no matter what the initial value y_{30} is. Thus the region of controllability is unbounded in the y_3 coordinate. Equation (4. 1. 1) shows directly that $|y_2| \to |d_2u/\lambda_2|$ as $t \to -\infty$, hence y_2 must be bounded by:

$$|y_2| < \frac{d_2u}{\lambda_2}$$

Numerically we have

$$|y_2| < 0.45 \qquad (4.1.3)$$

By (2. 2. 6)

$$y_2 = -\theta - \frac{1}{\lambda_2}\dot{\theta} + \frac{1}{g}\frac{\lambda_2\lambda_4}{\lambda_2-\lambda_4}$$

If only θ has non-zero initial value, it must be bound as:

$$|\theta(0)| < 0.45$$

This is the reason of using 0. 1 (radian) as the initial value of angle displacement.

4. 2 Analysis by Graphs

4. 2. 1 No control on Cart Position.

Equation (2. 1. 5) can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 17.2 & 0 & 0 & 3.5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -17.2 \\ 0 \\ 9.8 \end{bmatrix} \times u \qquad (2.1.5)$$

By (3. 1. 5) and (3. 2. 1. 1):

$$u = 0.24 \operatorname{sgn} \left[x_1 + 0.241\, x_2 + 0.138\, x_4 \right] \qquad (4.2.1.1)$$

Now, this system can be simulated. For better accuracy, we use sin x_1 instead of x_1. Graphs are shown in Appendix I. Same initial angle for all graphs.

Fig. I-1 shows pendulum angle vs. time, with initial angle displacement 0. 1 radian. It comes back very quickly.

Fig. I-2 shows the pendulum changing rate vs. time.

Fig. I-3 shows the cart position vs. time.

Fig. I-4 shows the phase plane ($\dot{\theta}$ vs. θ). The tracjectory does come back to the origin. It means stability.

Fig. I-5 shows $\dot{Z}$ vs. Z . As time goes on, the position rate decreases linearly to zero.

Fig. I-6 shows the control force vs. time. It is the bang-bang type force; only the direction of the force is switched by the relay.

4. 2. 2 With Control on Cart Position.

In this case, the system equations are same as before, but the control force changes as:

$$u = 0.24 \operatorname{sgn}\left[x_1 + 0.241x_2 + (0.102b + 0.138)x_4 + 0.204bx_3\right]$$

We simulate this system with the initial angle displacement as before (θ (o) = 0.1 radian). Graphs are listed in Appendix II.

Firstly with b = 0.1 for position control.

Fig.II-1 shows pendulum angle vs. time. It is stable.

Fig. II-2 shows the phase plane. The tracjectory goes to the origin.

Fig. II-3 shows the cart position vs. time. It does not come back very quickly, because the amount of b is too small.

Then take b = 0.7 for position control.

Fig. II-4 shows the stick angle vs. time.

Fig. II-5 shows the position vs. time. The cart comes back to where it started very quickly.

5. Optimal Discrete-time Control

5.1 The System in Discrete-time Form

Recall the original system as follows:

$$\dot{\underline{x}} = [A]\, \underline{x} + \underline{c}\, u \qquad (2.1.6)$$

Consider, firstly, the equation without control force.

$$\dot{\underline{x}} = [A]\, \underline{x} \qquad (5.1.1)$$

and assume a Taylor series

$$\underline{x}\,(t) = A_o + A_1 t + A_2 t^2 + \ldots + A_m t^m + \ldots \qquad (5.1.2)$$

to be the solution of the above homogeneous differential equation.

Then, set t = 0, one obtains:

$$\underline{x}\,(0) = A_o$$

Next, if (5.1.2) differentiated and then t set to zero, one obtains:

$$\dot{\underline{x}}\,(0) = A_1$$

But, from (5.1.1)

$$\dot{\underline{x}}\,(0) = [A]\, \underline{x}(0)$$

Then

$$A_1 = [A]\, \underline{x}\,(0)$$

If (5.1.2) is differentiated twice, and t set to zero, one obtaines:

$$\ddot{\underline{x}}\,(0) = 2A_2$$

or

$$2A_2 = \ddot{\underline{x}}\,(0) = [A]\, \dot{\underline{x}}\,(0) = [A]^2\, \underline{x}\,(0)$$

So that

$$A_2 = {}^1/_2 [A]^2 \, \underline{x}(0)$$

Continuing this process, all the terms A_i can be evaluated, and one obtains:

$$\underline{x}(t) = \left\{ [I] + [A]\, t + \frac{[A]^2 t^2}{2!} + \cdots + \frac{[A]^m t^m}{m!} + \cdots \right\} \underline{x}(0) \qquad (5.1.3)$$

By comparing it with the scalar expansion of e^{at}, it is obvious to have a more compact form, like:

$$\underline{x}(t) = e^{At} \, \underline{x}(0) \qquad (5.1.3.1)$$

In my case, e^{At} is a 4 x 4 matrix. It is usually called fundamental matrix and designated by:

$$\phi(t) \overset{D}{=} e^{At}$$

Apparently

$$\phi(-t) = e^{-At} = \frac{1}{\phi(t)} = \phi^{-1}(t) \qquad (5.1.4)$$

Also

$$\underline{x}(t) = \phi(t) \, \underline{x}(o)$$

Then

$$\dot{\underline{x}}(t) = \dot{\phi}(t) \, \underline{x}(o) = [A] \, \underline{x}(t) = [A] \, \phi(t) \, \underline{x}(o)$$

So that

$$\dot{\phi}(t) = [A] \, \phi(t) \qquad (5.1.5)$$

In order to solve the equation (2.1.6), we try to find a particular integral in the form of

$$\underline{x}_p(t) = \phi(t)\, y(t)$$

By putting into (2. 1. 6), one obtains

$$\dot{\phi}(t)\, y(t) + \phi(t)\, \dot{y}(t) = [A]\, \phi(t)\, y(t) + \underline{c}\, u(t)$$

By (5. 1. 5)

$$\phi(t)\, \dot{y}(t) = \underline{c}\, u(t)$$

$$y(t) = \int_0^t \phi^{-1}(\tau)\, \underline{c}\, u(\tau)\, d\tau$$

Then

$$\underline{x}_p(t) = \phi(t) \int_0^t \phi^{-1}(\tau)\, \underline{c}\, u(\tau)\, d\tau$$

By (5. 1. 4)

$$\underline{x}_p(t) = \phi(t) \int_0^t \phi(-\tau)\, \underline{c}\, u(\tau)\, d\tau \tag{5. 1. 6}$$

In evaluation of $\phi(t)$, we know the argument t represents the time interval between two instants. More conveniently, we can describe by:

$$\underline{x}(t_2) = \phi(t_2 - t_1)\, \underline{x}(t_1)$$

$$\underline{x}(t_3) = \phi(t_3 - t_2)\, \underline{x}(t_2) = \phi(t_3 - t_1)\, \underline{x}(t_1)$$

But $$\underline{x}(t_3) = \phi(t_3 - t_2)\, \phi(t_2 - t_1)\, \underline{x}(t_1)$$

$$\phi(t_3 - t_1) = \phi(t_3 - t_2) \cdot \phi(t_2 - t_1)$$

By this reason, (5. 1. 6) can be put into the more familiar form of a convolution integral.

$$x_p(t) = \int_0^t \phi(t - \tau)\, \underline{c}\, u(\tau)\, d\tau \tag{5. 1. 6. 1}$$

Then the general solution of (2. 1. 6) will be:

$$\underline{x}(t) = \phi(t)\,\underline{x}(o) + \int_0^t \phi(t-\tau)\,\underline{c}\,u(\tau)\,d\tau \qquad (5.1.7)$$

In case of discrete time, it turns out to be:

$$\underline{x}(k+1) = \phi(DT)\,\underline{x}(k) + \int_0^{DT} \phi(DT-\tau)\,\underline{c}\,u(\tau)\,d\tau \qquad (5.1.8)$$

Where

$DT \overset{D}{=}$ sampling time

By noting of a constant control force through the interval DT, one obtains:

$$\underline{x}(k+1) = \phi(DT)\,\underline{x}(k) + u(k)\cdot\int_0^{DT} \phi(DT-\tau)\,\underline{c}\,d\tau \qquad (5.1.9)$$

Or

$$\underline{x}(k+1) = \phi(DT)\cdot\,\underline{x}(k) + \underline{\Delta}\,u(k) \qquad (5.1.10)$$

Where

$$\underline{\Delta} \overset{D}{=} \int_0^{DT} \phi(DT-\tau)\,\underline{c}\,d\tau$$

$$\phi(DT) = [I] + [A]\cdot DT + \frac{[A]^2 (DT)^2}{2!} + \ldots + \frac{[A]^m (DT)^m}{m!} + \ldots$$

The computation of $\underline{\Delta}$ (DT) and ϕ (DT) would be very tedious, but, with the high speed computer, they can be obtained within seconds. The program to achieve this is shown in Appendix IV-3. (DT = 0.1 sec.)

Where

$$\underline{\Delta} = \begin{bmatrix} -0.0817 \\ -1.6067 \\ 0.0458 \\ 0.8882 \end{bmatrix}$$

$$\phi = \begin{bmatrix} 1.0872 & 0.1029 & 0.0000 & 0.0166 \\ 1.7697 & 1.0872 & 0.0000 & 0.3268 \\ 0.0000 & 0.0000 & 1.0000 & 0.0906 \\ 0.0000 & 0.0000 & 0.0000 & 0.8187 \end{bmatrix}$$

5.2 Evaluation of Control Force.

In optimal discrete-time control, if we want to minimize the following cost function.

$$J(n) = \sum_{k=1}^{n} \left[\underline{x}^t (k)\, Q\, \underline{x} (k) + ru^2 (k-1) \right] \qquad (5.2.1)$$

The second term represents the amount of control force which can be allowed, arbitrarily r set to 1. The first term gives the choice of state variables which will be minimized. In my case, x_1 and x_3 are those variables. So, Q becomes:

$$[Q] = \begin{bmatrix} 1 & & 0 & \\ & 0 & & \\ 0 & & 1 & \\ & & & 0 \end{bmatrix}$$

By (5.1.10) (5.2.1)

$$J(n) = \left\{ \phi x(n-1) + \Delta u(n-1) \right\}^t Q \left\{ \phi x(n-1) + \Delta u(n-1) \right\} + ru^2(n-1) + J(n-1)$$

Minimizing with respect to u(n-1), $\frac{\partial J(n)}{\partial u(n-1)} = 0$ and noticing J(n-1) is independent of u(n-1), gives

$$\left\{ x^t (n-1)\, \phi^t + u(n-1) \Delta^t \right\} Q\Delta + \Delta^t Q \left\{ \phi\, x (n-1) + \Delta u(n-1) \right\} + 2ru(n-1) = 0$$

$$u(n-1) = -\frac{\Delta^t Q \phi}{\Delta^t Q \Delta + r}\, \underline{x} (n-1) \qquad (5.2.2)$$

Define

$$a_1^t \overset{D}{=} - \frac{\Delta^t Q \phi}{\Delta^t Q \Delta + r} \tag{5.2.3}$$

Then

$$u(n-1) = a_1^t \underline{x}(n-1) \tag{5.2.2.1}$$

Substitute u(n-1) to (5.2.1), one obtains: (5.2.4)

$$J(n) = \left\{\phi\left[\phi x(n-2)+\Delta u(n-2)\right] + \Delta a_1^t[\;\;]\right\}^t Q \left\{\phi[\;\;] + \Delta a_1^t[\;\;]\right\}$$

$$+ r[\;\;]^t a_1 a_1^t [\;\;] + [\;\;]^t Q[\;\;] + ru^2(n-2) + J(n-2)$$

Where $a_1 = (a_1^t)^t$

$$[\;\;] \overset{D}{=} \left[\phi x(n-2) + \Delta u(n-2)\right]$$

Further defining

$$\psi_1 \overset{D}{=} \phi + \Delta a_1^t \tag{5.2.5}$$

The first and third terms of (5.2.4) can be combined as:

$$\left\{[\;\;]^t \phi^t + [\;\;]^t a_1 \Delta^t\right\} Q \left\{\phi[\;\;] + \Delta a_1^t[\;\;]\right\} + [\;\;]^t Q[\;\;]$$

$$= [\;\;]^t(\phi^t Q \phi + \phi^t Q \Delta a_1^t + a_1 \Delta^t Q \phi + a_1 \Delta^t Q \Delta a_1^t)[\;\;] + [\;\;]^t Q[\;\;]$$

$$= [\;\;]^t(\phi^t Q\langle \phi + \Delta a_1^t\rangle + a_1 \Delta^t Q\langle \phi + \Delta a_1^t\rangle)[\;\;] + [\;\;]^t Q[\;\;]$$

$$= [\;\;]^t(\phi^t + a_1\Delta^t) Q (\phi + \Delta a_1^t)[\;\;] + [\;\;]^t Q[\;\;]$$

$$= [\;\;]^t(\psi_1^t Q \psi_1)[\;\;] + [\;\;]^t Q[\;\;]$$

$$= [\;\;]^t(\psi_1^t Q \psi_1 + Q)[\;\;]$$

Now, (5.2.4) becomes:

$$J(n) = [\;\;]^t(\psi_1^t Q \psi_1 + Q)[\;\;] + r[\;\;]^t a_1 a_1^t [\;\;] + ru^2(n-2) + J(n-2)$$

$$= [\;\;]^t(\psi_1^t Q \psi_1 + Q + r a_1 a_1^t)[\;\;] + ru^2(n-2) + J(n-2)$$

Define

$$p_1 \overset{D}{=} \psi_1^t Q \psi_1 + Q + r a_1 a_1^t \qquad (5.2.6)$$

$\frac{\partial J(n)}{\partial u(n-2)} = 0$ gives:

$$[\quad]^t p_1 \Delta + \Delta^t p_1 [\quad] + 2\,ru(n-2) = 0$$

$$[x^t(n-2)\phi^t + u(n-2)\Delta^t] p_1 \Delta + \Delta^t p_1 [\quad] + 2\,ru(n-2) = 0$$

$$u(n-2) = -\frac{\Delta^t p_1 \phi}{\Delta^t p_1 \Delta + r}\, \underline{x}(n-2) \qquad (5.2.7)$$

Define

$$a_2^t = -\frac{\Delta^t p_1 \phi}{\Delta^t p_1 \Delta + r} \qquad (5.2.8)$$

Then

$$u(n-2) = a_2^t\, \underline{x}(n-2) \qquad (5.2.7.1)$$

Define $\quad p_o \overset{D}{=} Q$

(5. 2. 3) becomes:

$$a_1^t = -\frac{\Delta^t p_o \phi}{\Delta^t p_o \Delta + r} \qquad (5.2.3.1)$$

Continuing one more stage, and set $\frac{\partial J(n)}{\partial u(n-3)} = 0$, one obtains:

$$\psi_2 = \phi + \Delta a_2^t$$

$$p_2 = \psi_2^t p_1 \psi_2 + Q + r a_2 a_2^t$$

$$a_3^t = -\frac{\Delta^t p_2 \phi}{\Delta^t p_2 \Delta + r}$$

$$u(n-3) = a_3^t\, \underline{x}(n-3)$$

Continuing on the same procedure, one can expect the following general forms.

$$\psi_k = \phi + \Delta a_k^t$$

$$p_k = \psi_k^t \, p_{k-1} \, \psi_k + Q + r \, a_k a_k^t$$

$$a_k^t = -\frac{\Delta^t p_{k-1} \phi}{\Delta^t p_{k-1} \Delta + r}$$

$$u(n-k) = a_k^t \, \underline{x} \, (n-k)$$

We note that u(n-k) depends solely on those present states $\underline{x}$ (n-k). Thus makes clear Bellman's "Principle of Optimality", which states: "An optimal policy has the property that whatever the initial state and the initial control input rector are, the remaining control input rectors must constitute an optimal policy with regard to the state resulting from the first control signal."

When the number of stages gets very large, a_k^t converges to some final set of values. Thus exists some fixed values for the feedback compensation for all values of time and state variables.

With the help of CDC 1604 computer, 200 stages are accomplished. The program is shown in Appendix IV-4. Finally, one obtains, for all sampling time:

$$u(t) = \begin{bmatrix} 2.4846 \\ 0.5990 \\ 0.0074 \\ 0.3448 \end{bmatrix} \underline{x} \, (t) \qquad (5.2.9)$$

5.3 Simulation by Integration

With the control force shown in (5.2.9), we can simulate the system by calling a subcontine for integration. The control force is calculated during every sampling instant and hold constant until the next sampling instant.

The FORTRAN program achieving this purpose is shown in Appendix IV-5. Initial angle displacement is same as before, $x(1) = 0.1$ radian. Sampling time $DT = 0.1$ second.

The graphs are collected in Appendix III.

Fig. III-1 shows stick angle vs. time. The angle comes back after about 40 stages.

Fig. III-2 shows phase plane.

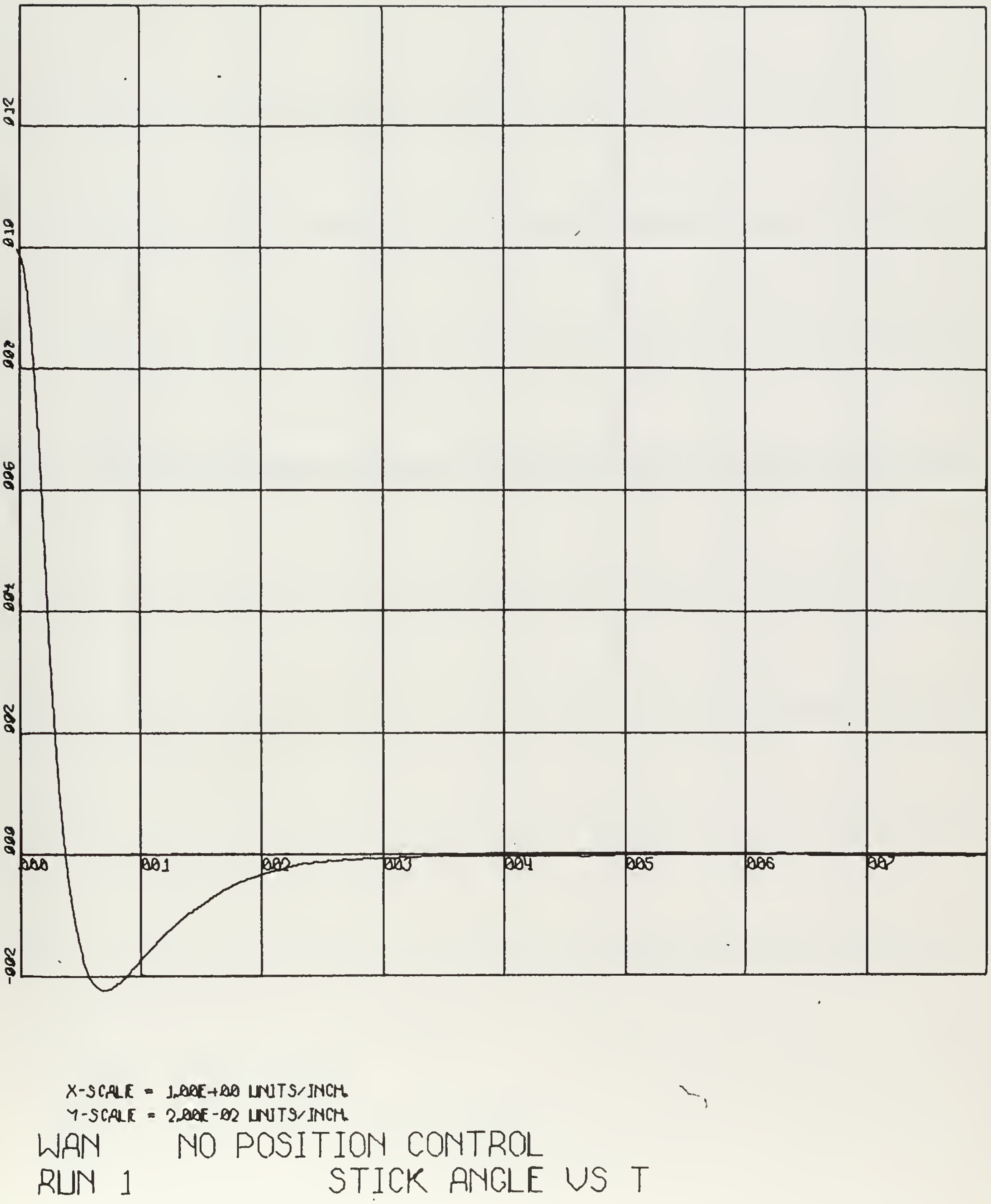
Fig. III-3 shows control force vs. time.

Fig. III-4 shows cart position vs. time. It gets back to the starting position comparatively slowly.

APPENDIX I

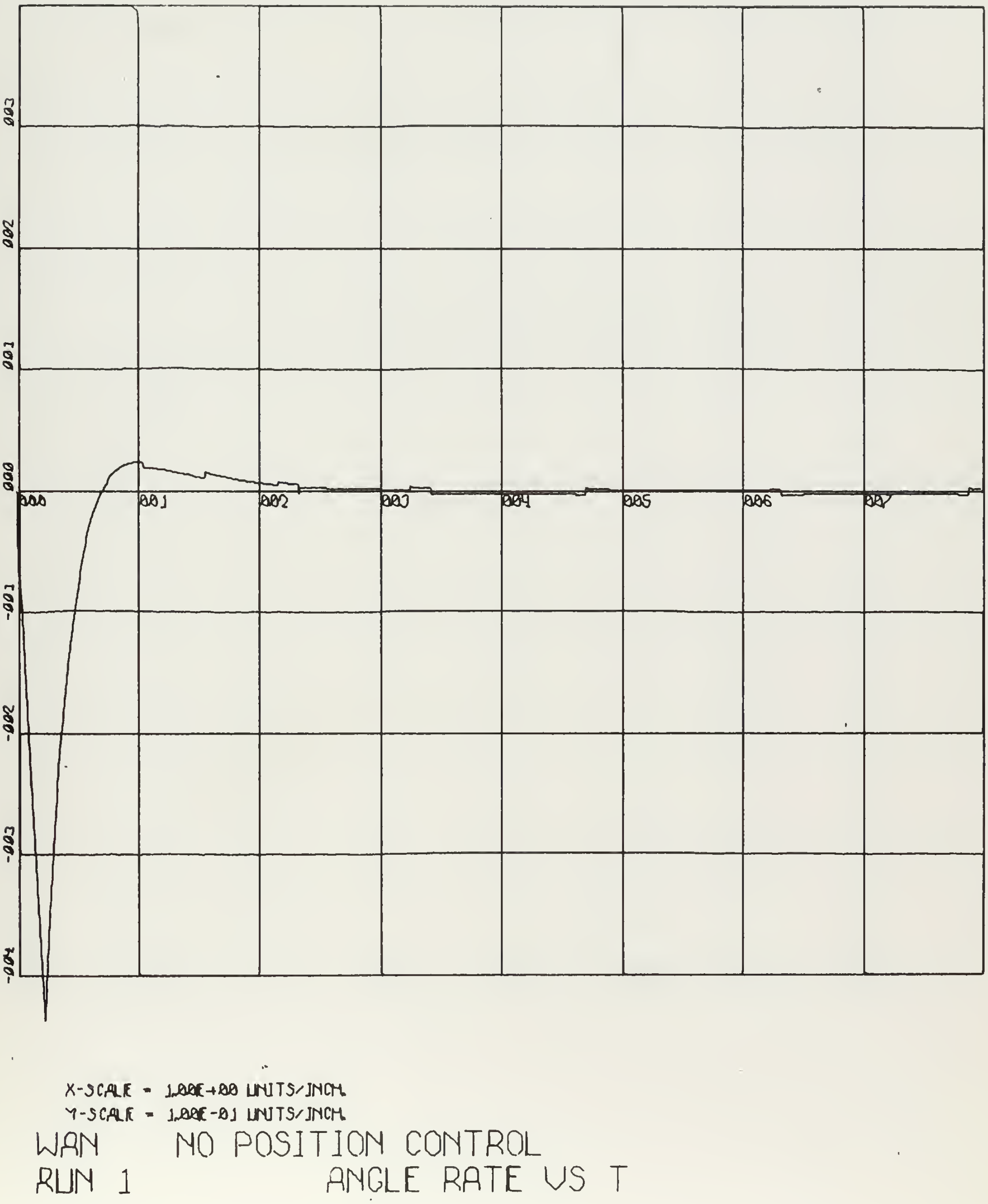
Graphs for No Control of Cart Position

1. Stick Angle vs. Time Without Position Control
2. Stick Angle Changing Rate vs. Time Without Position Control
3. Postion vs. Time Without Position Control
4. Phase Plane Without Position Control
5. Position Changing Rate vs. Position Without Position Control
6. Control Force vs. Time Without Position Control



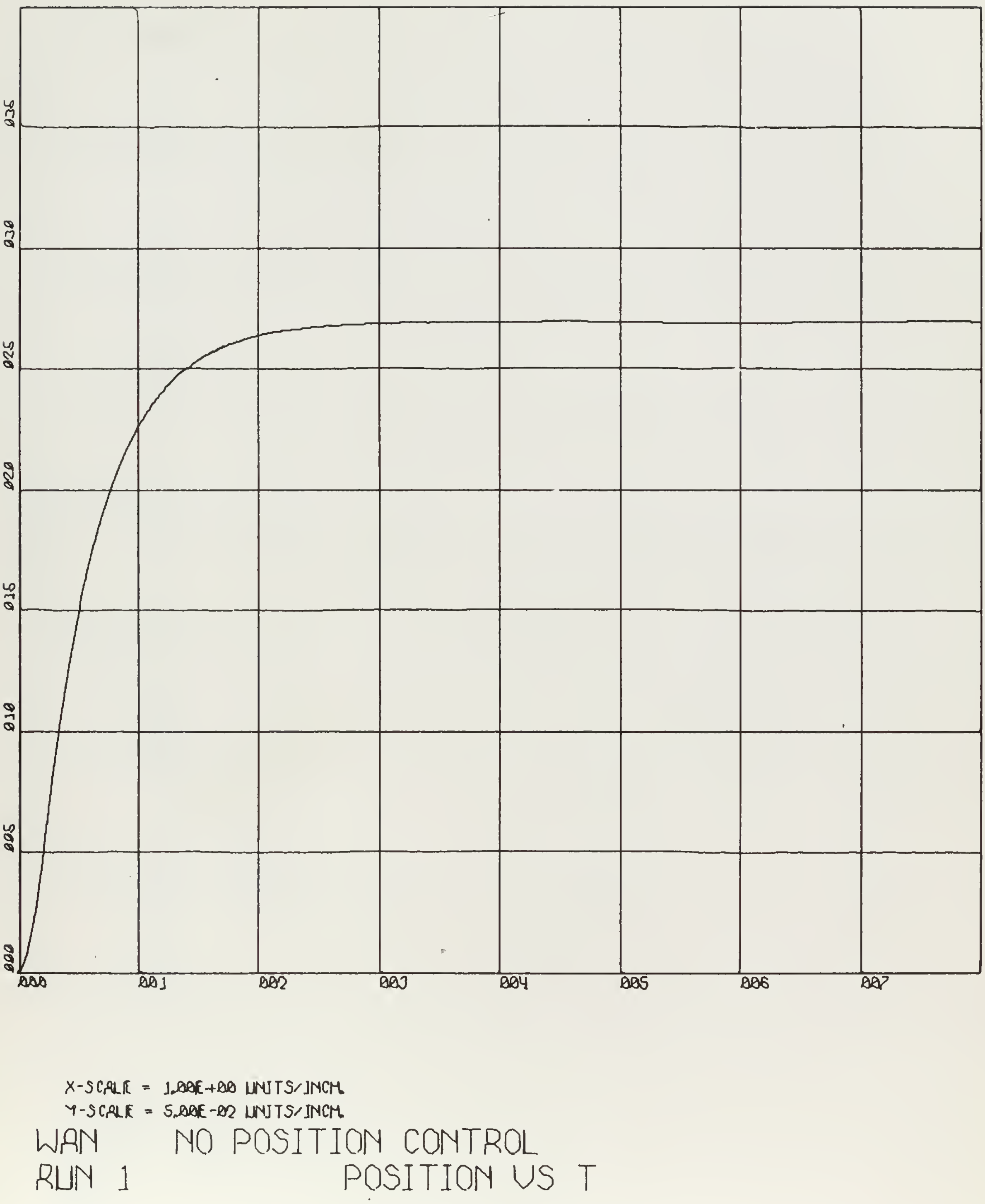
X-Scale: 1.0 Second/Unit
Y-Scale: 0.02 Radian/Unit

FIG. (I-1). Stick Angle Vs. Time Without Position Control



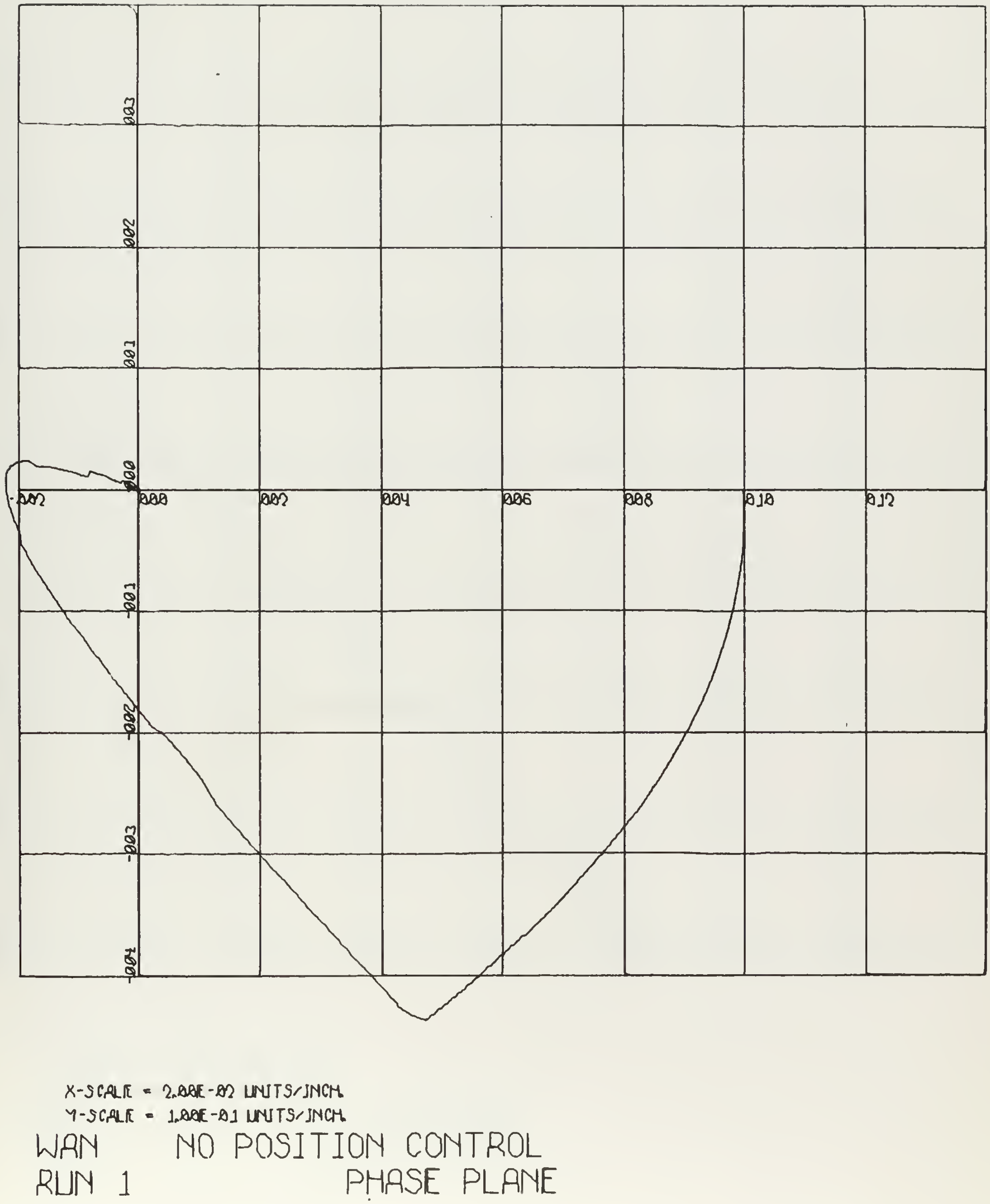
X Scale: 1. 0 Second/Unit
Y Scale: 0. 1 Radian per Second/Unit

FIG. - (I-2). Stick Angle Changing Rate Vs. Time Without Position Control



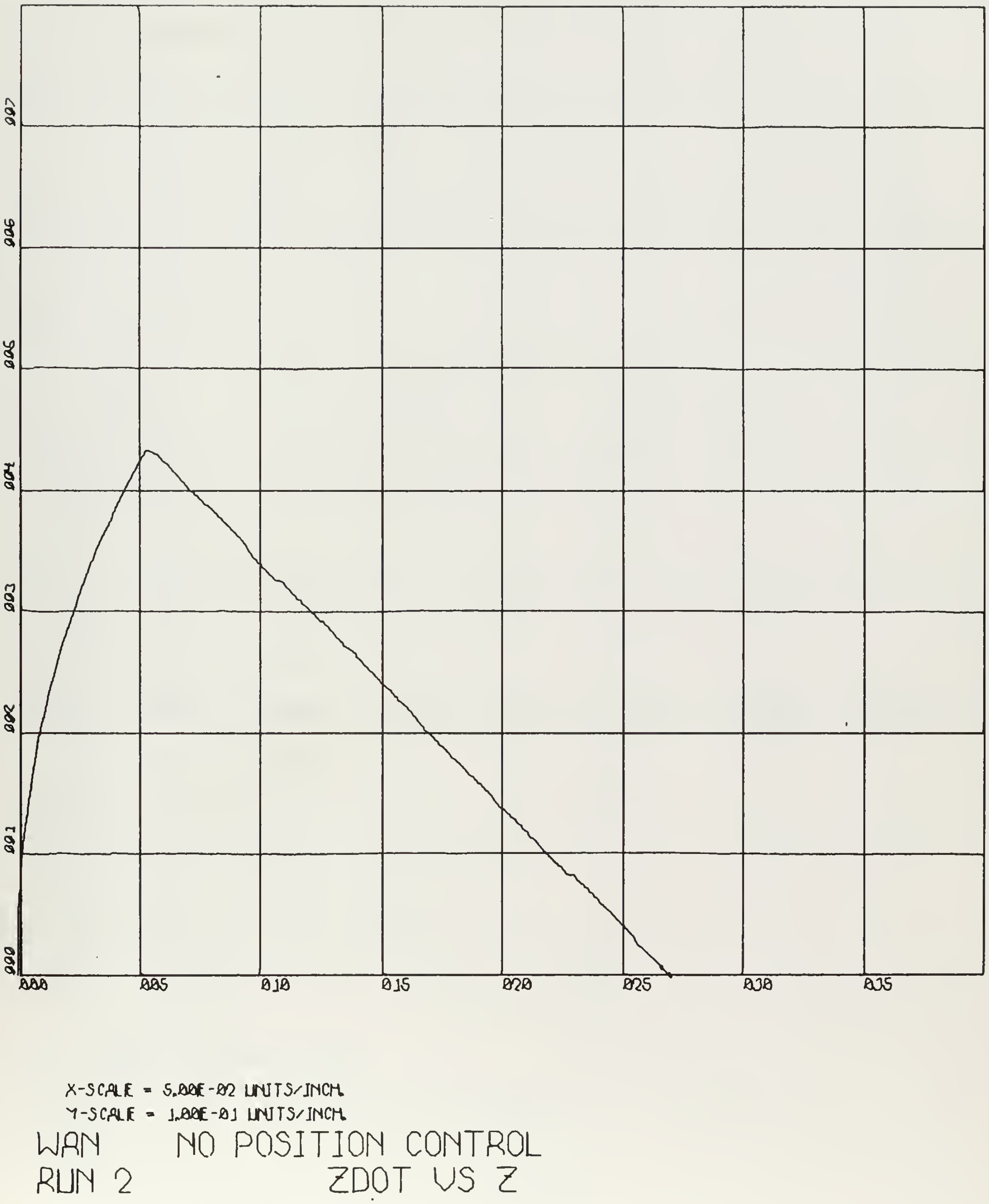
X Scale: 1.0 Second/Unit
Y Scale: 0.05 Meter/Unit

FIG. (I-3). Position Vs. Time Without Position Control



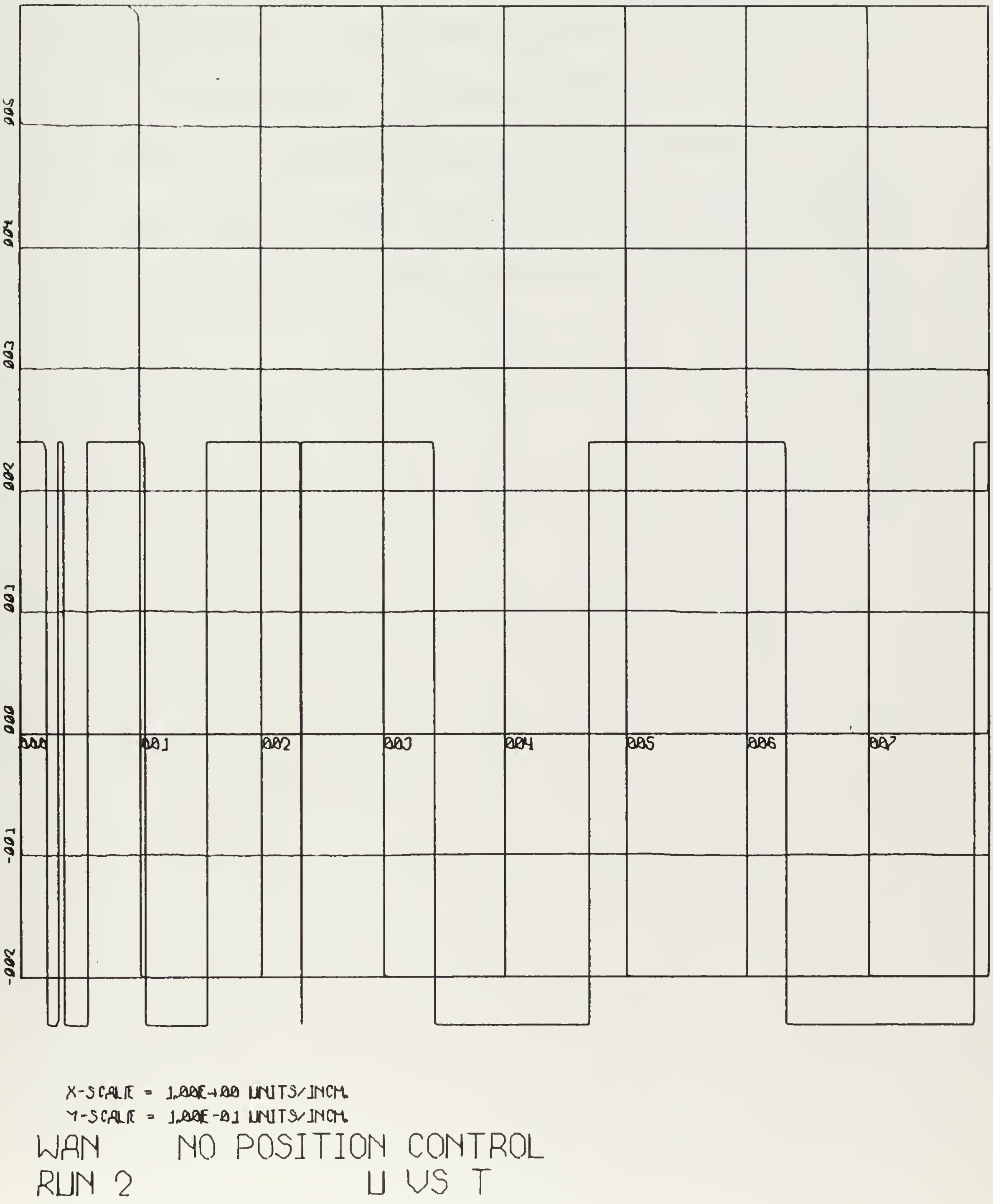
X Scale: 0.02 Radian/Unit
Y Scale: 0.1 Radian per Second/Unit

FIG. (I-4). Phase Plane Without Position Control



X Scale: 0.05 Meter/Unit
Y Scale: 0.1 Meter per Second/Unit

FIG. (I-5). Position Changing Rate Vs. Position Without Position Control



Note : Control force has no dimension.

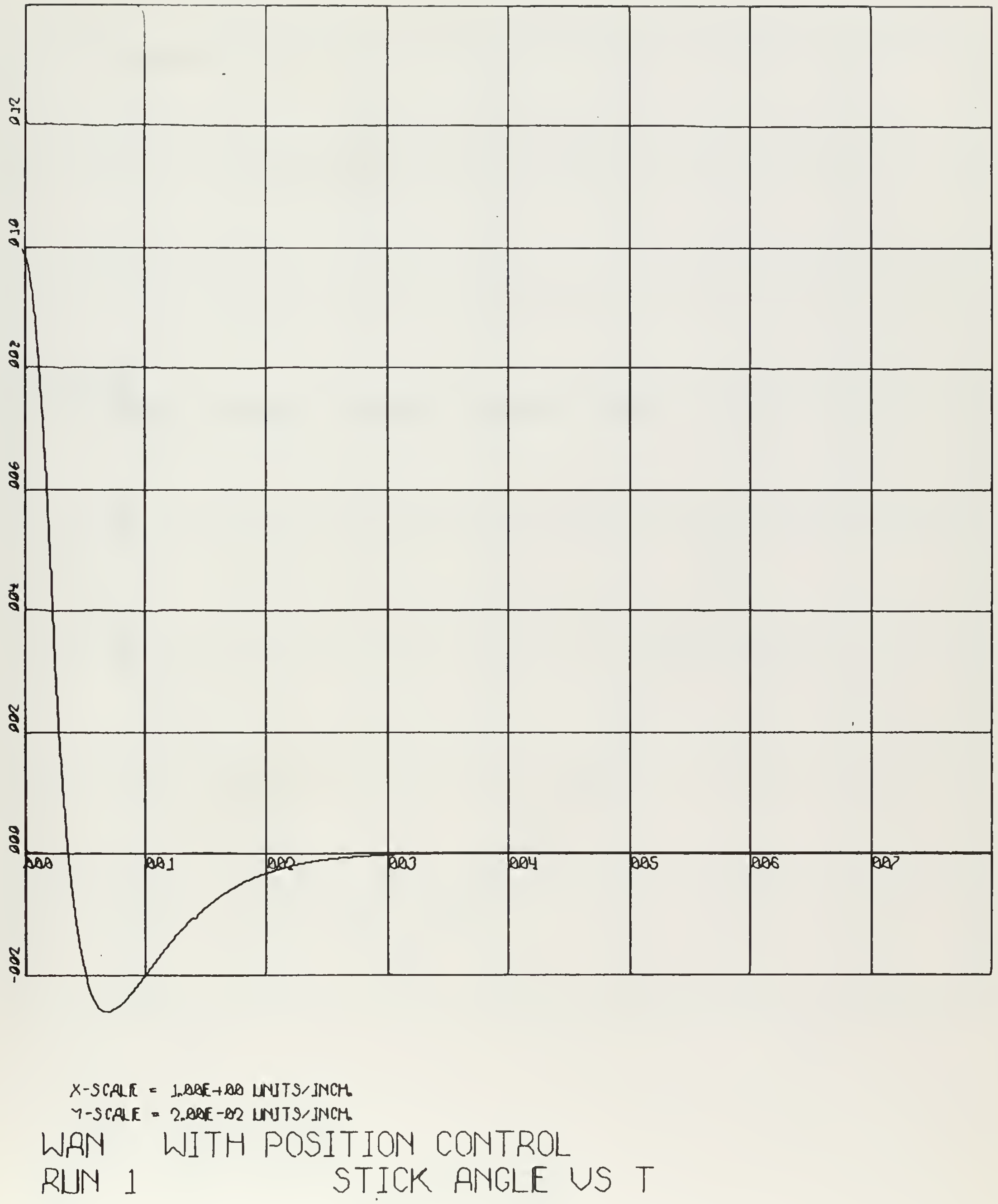
X Scale: 1. 0 Second/Unit
Y Scale: 0. 1 Unit/Unit

FIG. (I-6). Control Force Vs. Time Without Position Control

APPENDIX II

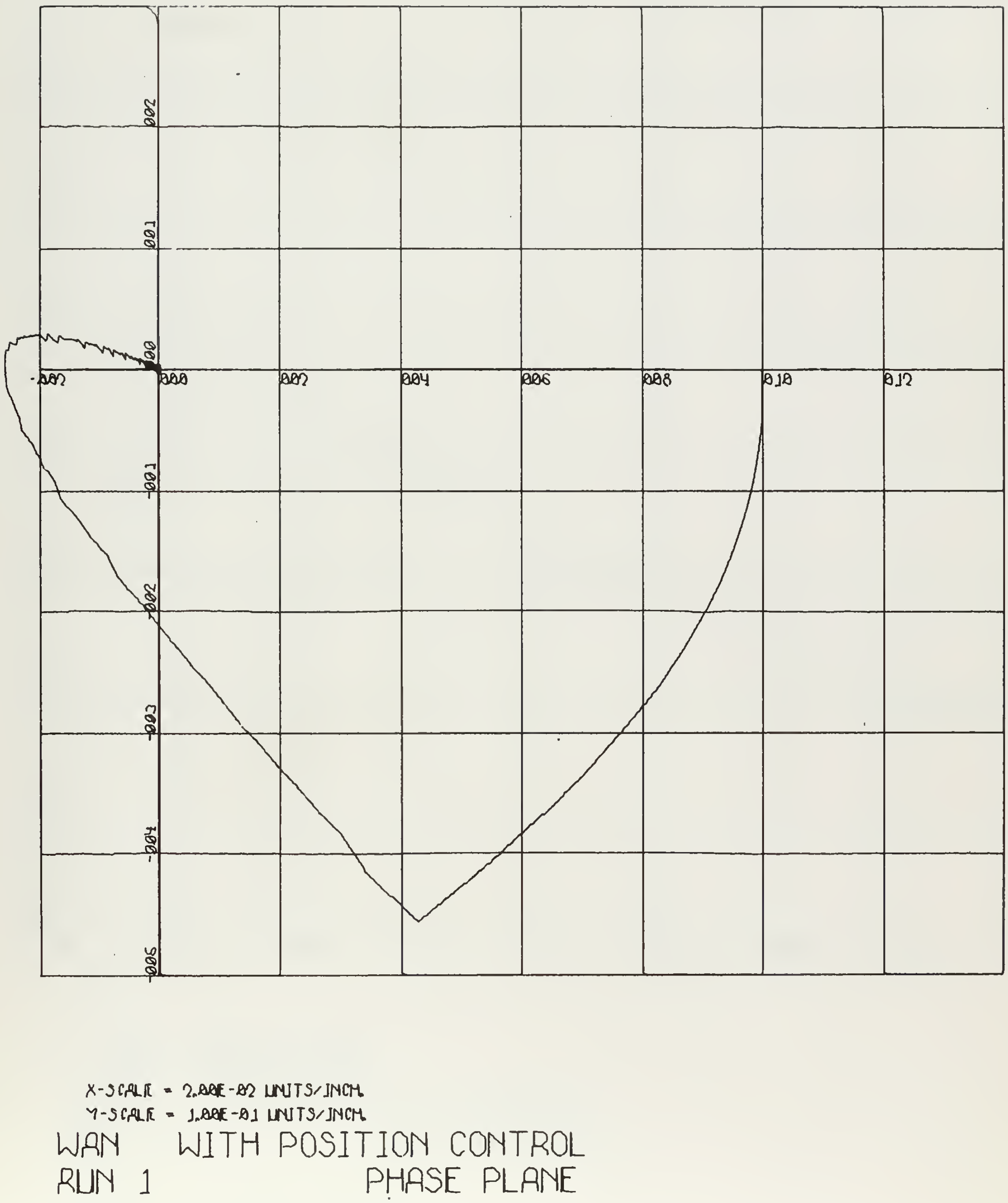
Graphs for Control of Cart Position

1. Stick Angle Vs. Time With Less Position Control (b = 0.1)
2. Phase Plane With Less Position Control (b = 0.1)
3. Position Vs. Time With Less Position Control (b = 0.1)
4. Stick Angle Vs. Time With More Position Control (b = 0.7)
5. Position Vs. Time With More Position Control (b = 0.7)



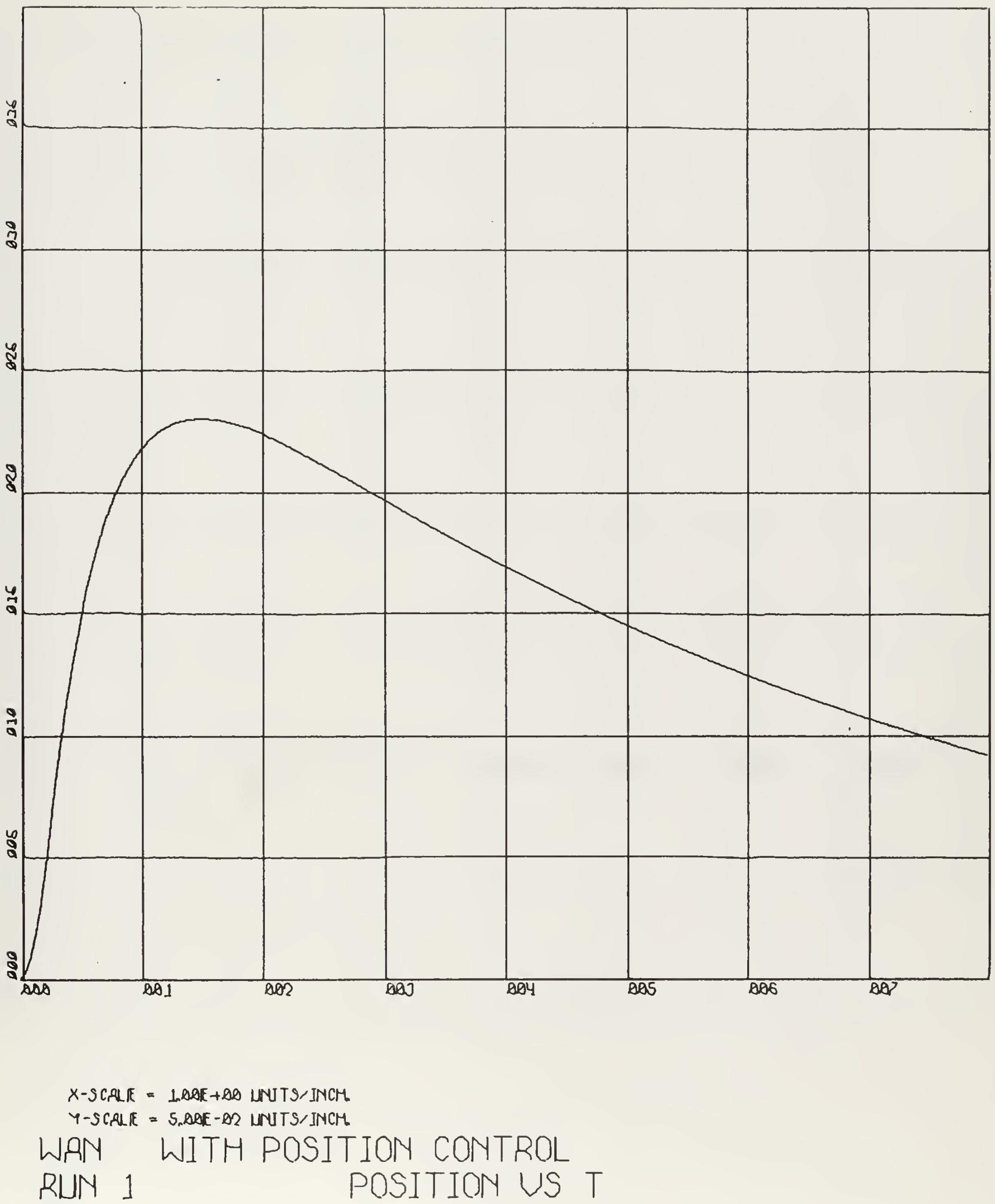
X Scale: 1. 0 Second/Unit
Y Scale: 0. 02 Radian/Unit

FIG. (II-1). Stick Angle Vs. Time With Less Position Control (b=0. 1)



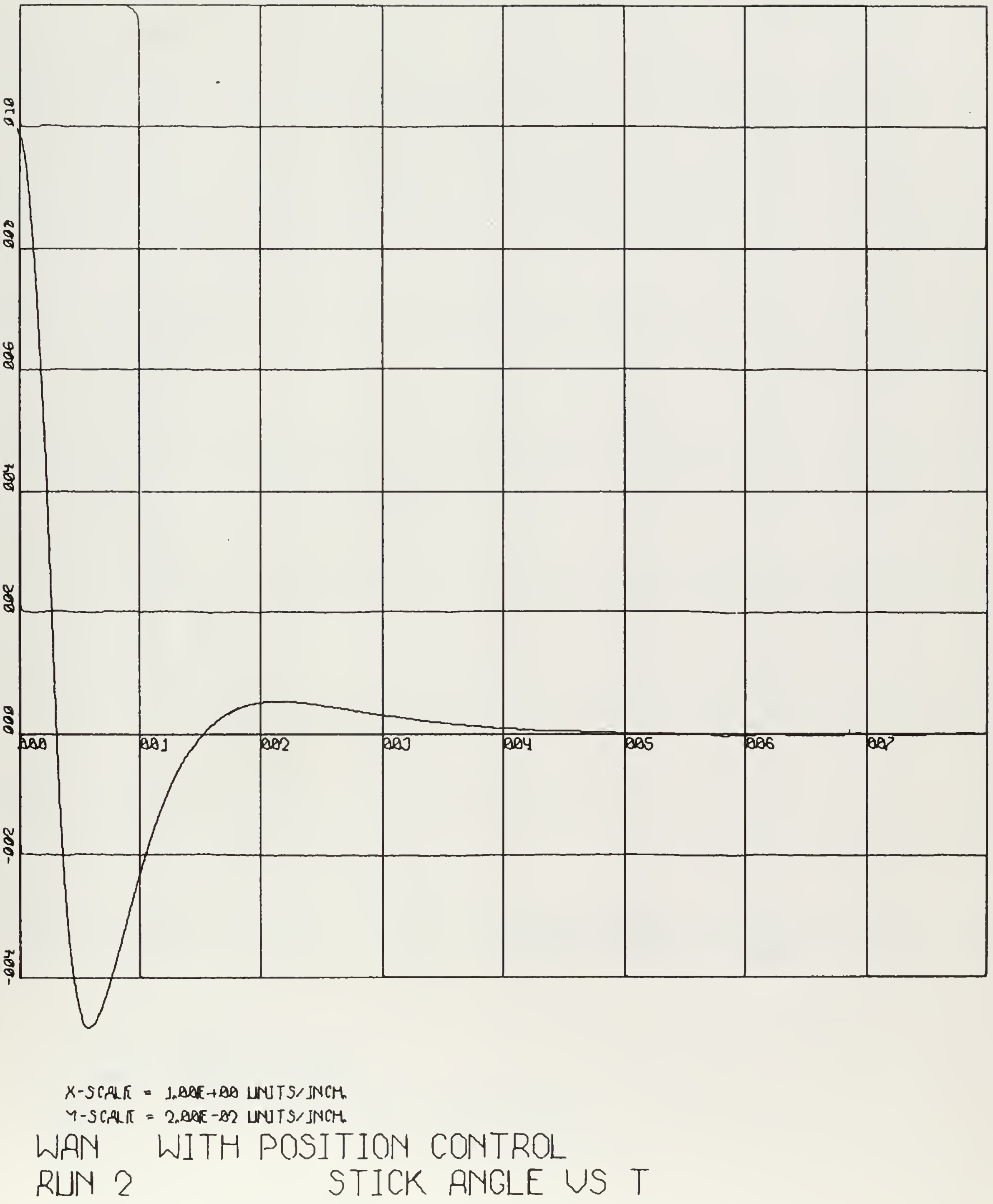
X Scale: 0.02 Radian/Unit
Y Scale: 0.1 Radian per Second/Unit

FIG. (II-2). Phase Plane With Less Position Control (b=0.1)



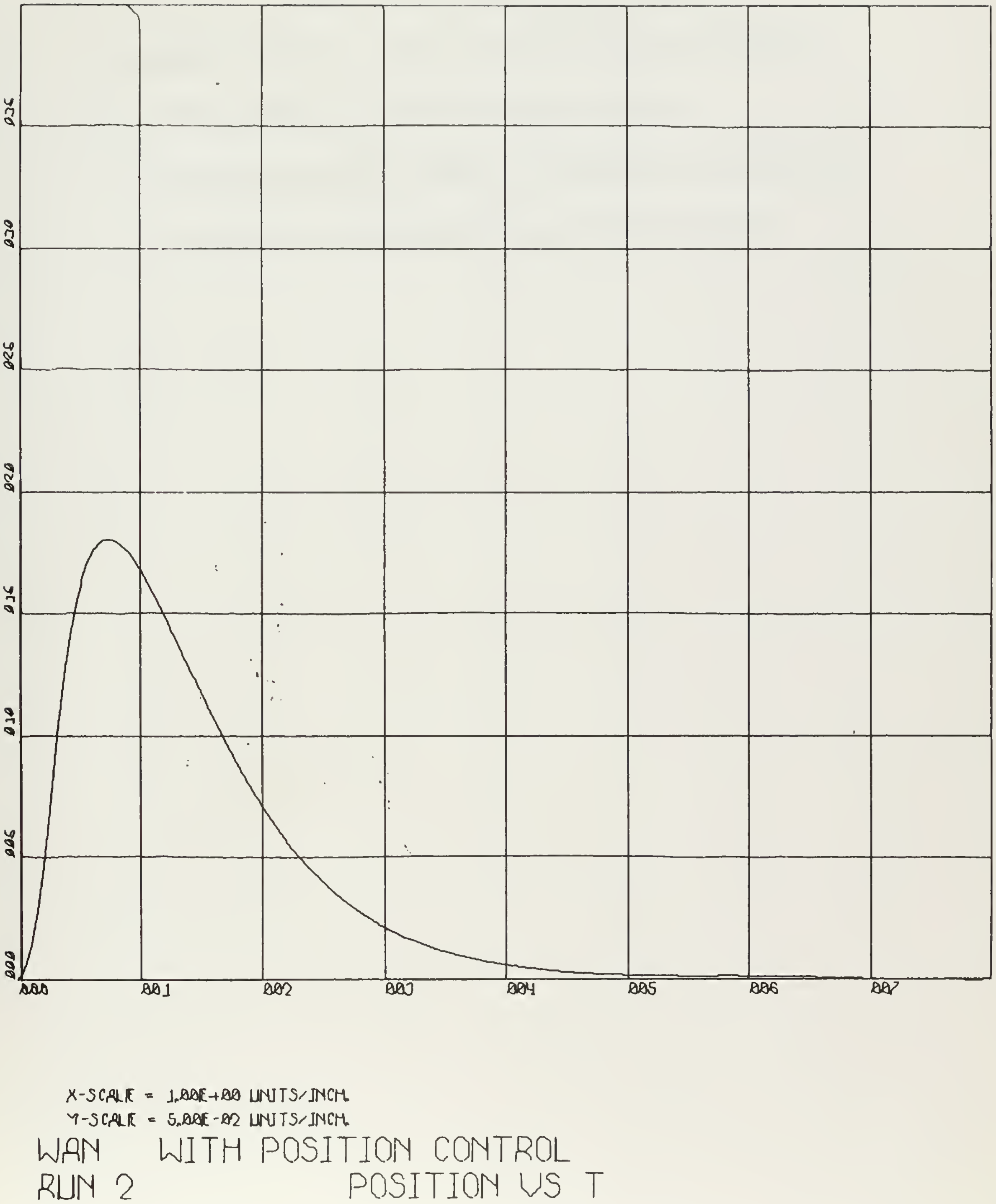
X Scale: 1.0 Second/Unit
Y Scale: 0.05 Meter/Unit

FIG. (II-3). Position Vs. Time With Less Position Control (b = 0.1)



X Scale: 1.0 Second/Unit
Y Scale: 0.02 Radian/Unit

FIG. (II-4). Stick Angle Vs. Time With More Position Control (b=0.7)



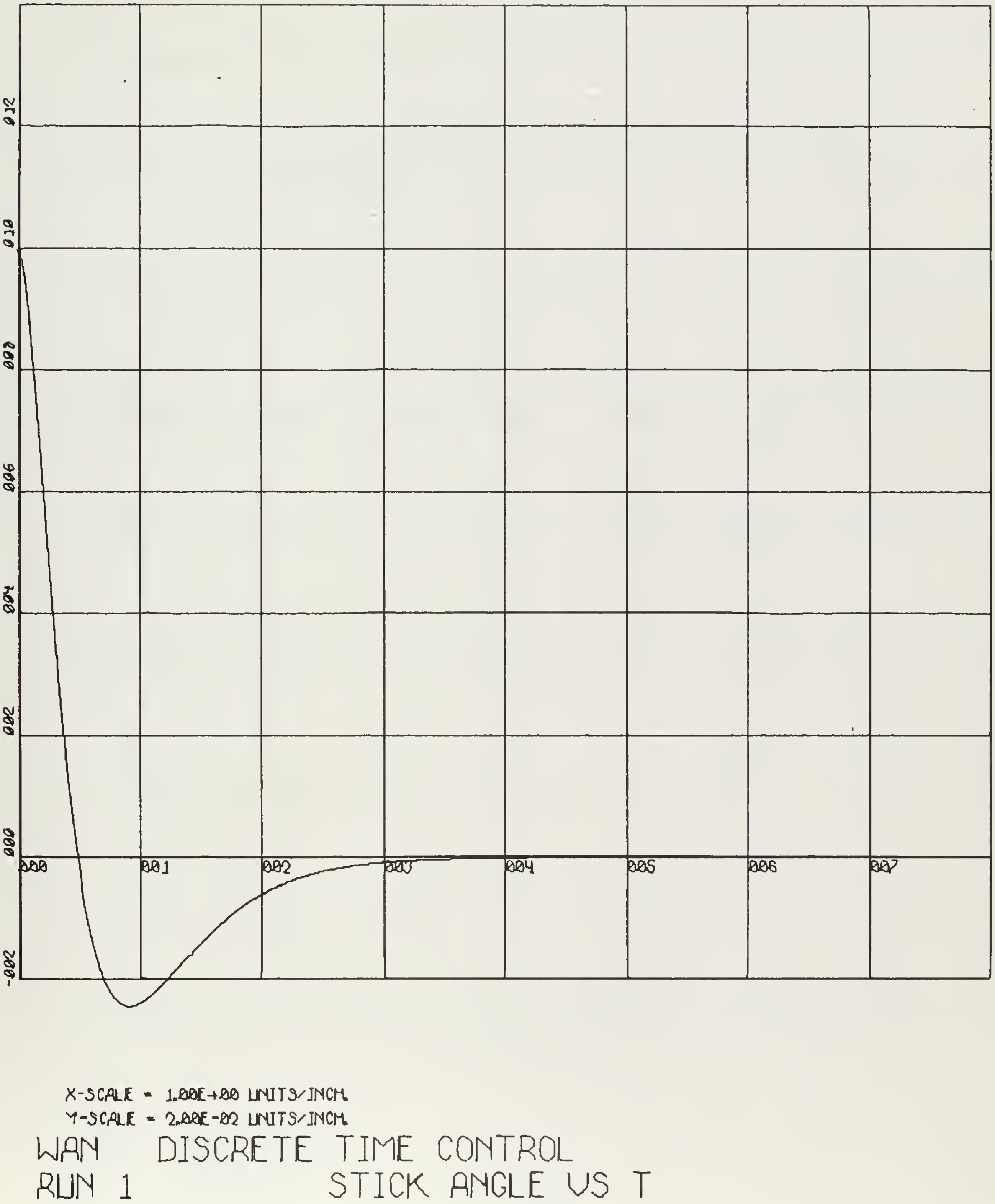
X Scale: 1.0 Second/Unit
Y Scale: 0.05 Meter/Unit

FIG. (II-5). Position Vs. Time With More Position Control (b = 0.7)

APPENDIX III

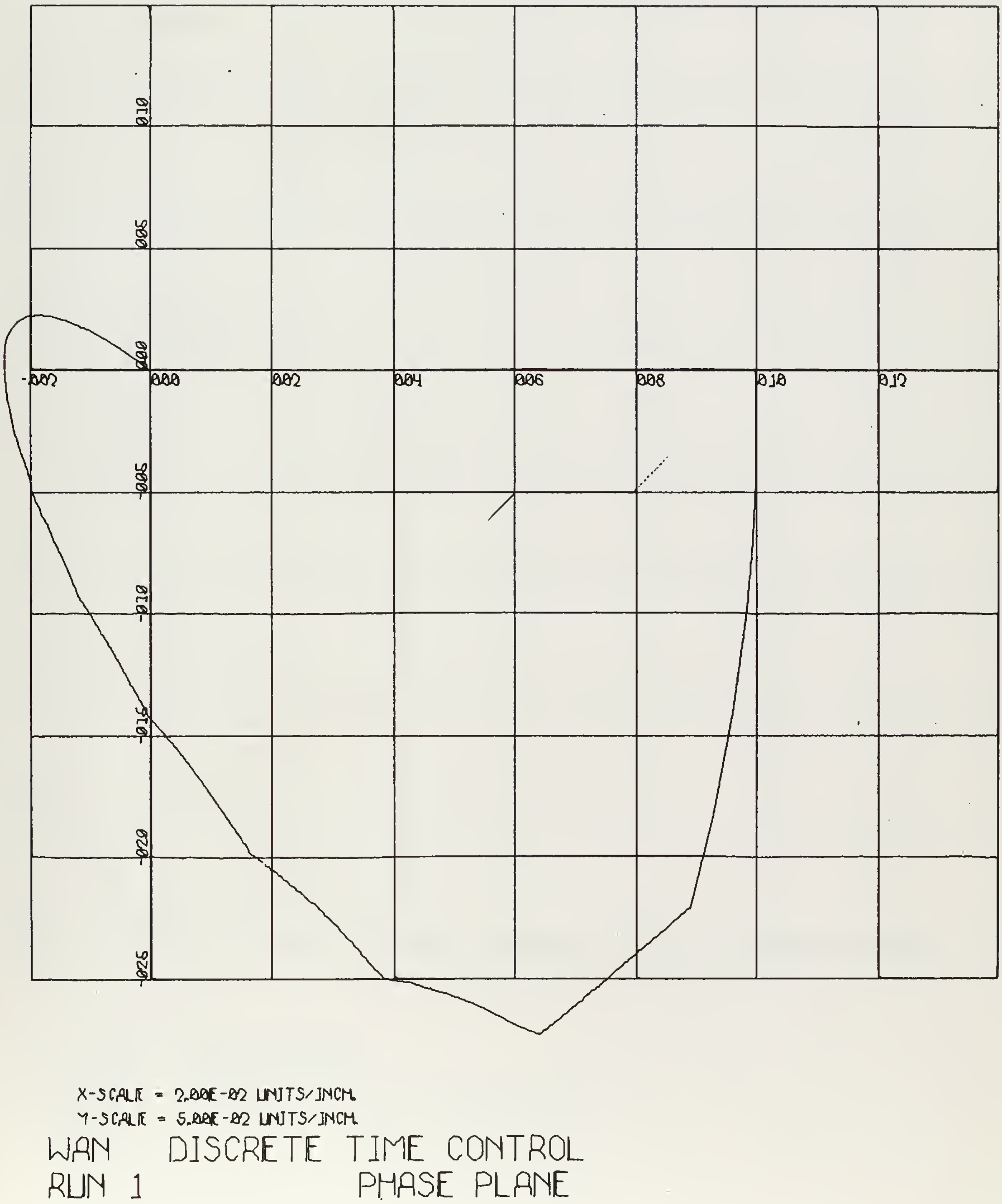
Graphs for Discrete-time Control

1. Stick Angle Vs. Time for Discrete-time Control
2. Phase Plane for Discrete-time Control
3. Control Force Vs. Time for Discrete-time Control
4. Position Vs. Time for Discrete-time Control



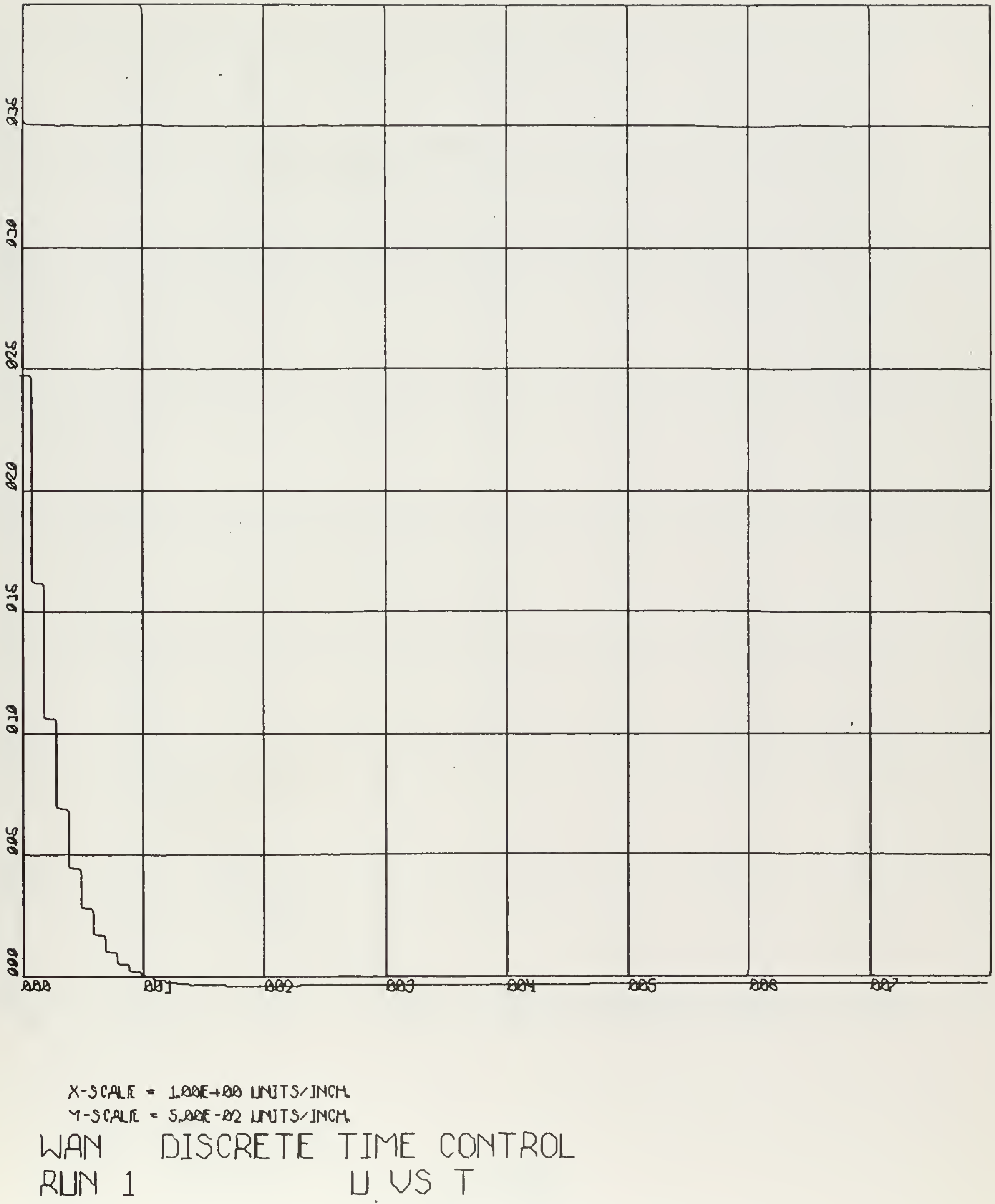
X Scale: 1.0 Second/Unit
Y Scale: 0.02 Radian/Unit

FIG. (III-1). Stick Angle Vs. Time for Discrete-time Control



X Scale: 0. 02 Radian/Unit
Y Scale: 0. 05 Radian per Second/Unit

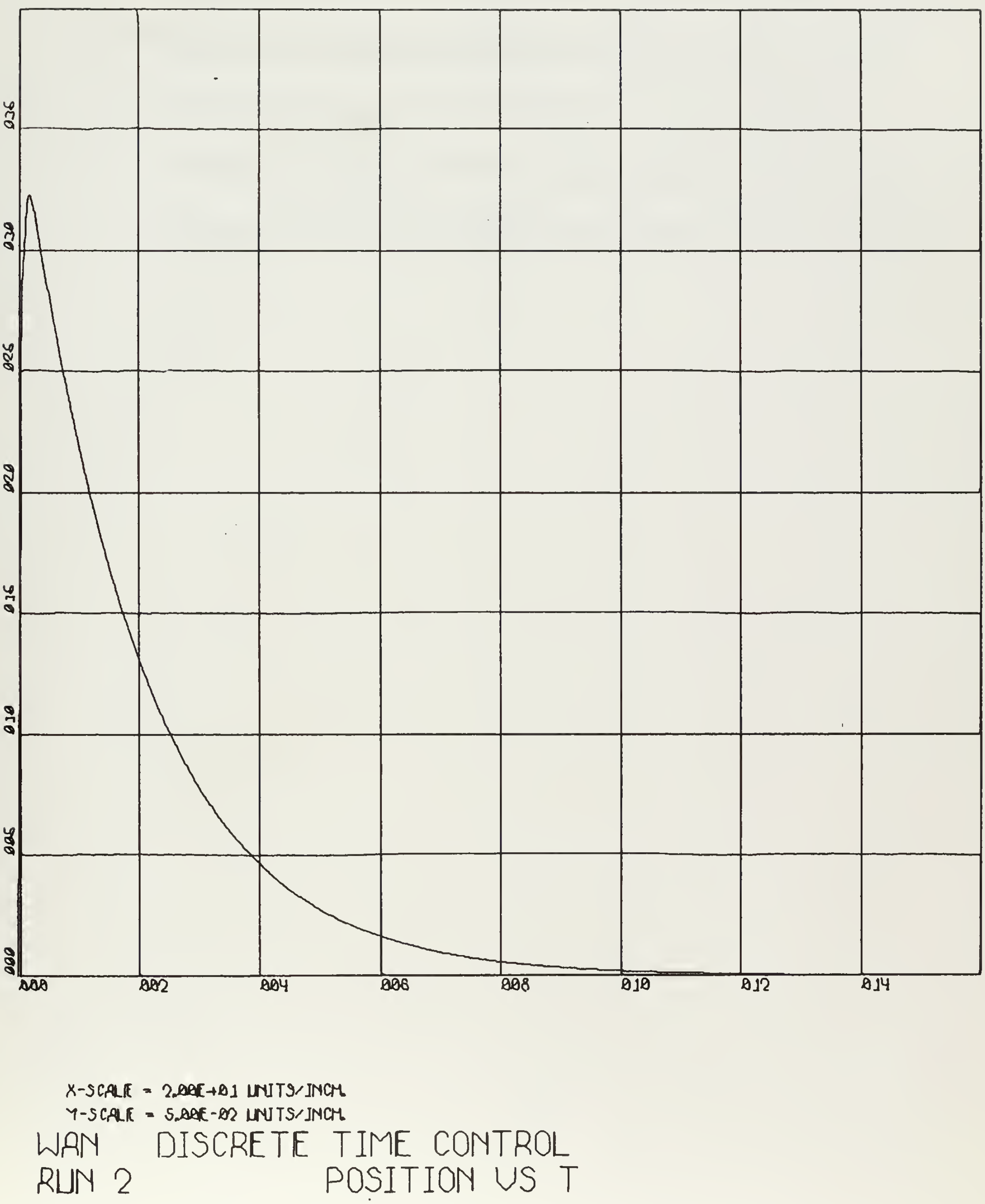
FIG. (III-2). Phase Plane for Discrete-time Control



Note: Control force has no dimension.

X Scale: 1.0 Second/Unit
Y Scale: 0.05 Unit/Unit

FIG. (III-3). Control Force Vs. Time for Discrete-time Control



X Scale: 20.0 Second/Unit
Y Scale: 0.05 Meter/Unit

FIG. (III-4). Position Vs. Time for Discrete-time Control

APPENDIX IV

FORTRAN Programs

1. FORTRAN Program for Simulation without Position Control
2. FORTRAN Program for Simulation with Position Control
3. FORTRAN Program for Calculating ϕ and Δ Matrix
4. FORTRAN Program for Calculating Discrete-time Control Force
5. FORTRAN Program for Simulation of Discrete-time Control

```
..JOB0250F,WAN
      PROGRAM BROOM
C     NO CART POSITION CONTROL
      DIMENSION X(30),XDOT(30),C(15)
      C(10)=1.0
    1 CALL INTEG1 (T,X,XDOT,C)
      XDOT(1)=X(2)
      XDOT(2)=17.2*SINF(X(1))+3.5*X(4)-17.2*U
      XDOT(3)=X(4)
      XDOT(4)=-2.0*X(4)+9.8*U
      U=0.24*SIGNF(1.,X(1)+C(1)*X(2)+C(2)*X(4))
      X(5)=U
      GO TO 1
      END
      END
```

```
      PROGRAM BROOM
C     POSITION CONTROL
C     RUN NO. TWO FOR MORE Y3 FEED BACK
      DIMENSION X(30),XDOT(30),C(15)
      C(10)=1.
    1 CALL INTEG1 (T,X,XDOT,C)
      XDOT(1)=X(2)
      XDOT(2)=17.2*SINF(X(1))+3.5*X(4)-17.2*U
      XDOT(3)=X(4)
      XDOT(4)=-2.*X(4)+9.8*U
      U=0.24*SIGNF(1.,X(1)+C(1)*X(2)+(C(3)*0.102+C(2))
     1*X(4)+C(3)*0.204*X(3))
      X(5)=U
      GO TO 1
      END
      END
```

```
..JOB0250F,WAN
      PROGRAM PHIDEL
C     COMPUTING PHI AND DELTA MATRIX
C     DT=SAMPLING TIME
      DIMENSION A(12,12),PHI(12,12),TERM(12,12),WORM(12,12)
     1,DEL(4),DELM(4,4),TELM(4,4),DELP(4,4),C(4)
      N=4
      DT=0.1
      TM=0.0
      READ1,((A(IR,IC),IC=1,N),IR=1,N)
    1 FORMAT ((4F20.8))
      READ 1,(C(I),I=1,N)
 1003 PRINT 399,DT,((A(IR,IC),IC=1,N),IR=1,N)
  399 FORMAT(///1X,3HDT=,1F5.3///,1X,7HA(I,J)=/,((4F10.2)))
      PRINT 3991 (C(I),I=1,N)
 3991 FORMAT(///1X,5HC(I)=/(4F10.2))
C
      DO 400 IR=1,N
      DO 400 IC=1,N
      TERM(IR,IC)=0.0
      WORM(IR,IC)=0.0
      TERM(IR,IR)=1.0
      TELM(IR,IC)=TERM(IR,IC)*DT
      DELP(IR,IC)=TELM(IR,IC)
      DELM(IR,IC)=0.0
      DEL(IR)=0.
  400 PHI(IR,IC)=TERM(IR,IC)
C
    4 TM=1.0+TM
      DO 500   IR=1,N
      DO 500   IC=1,N
      DO 500   JN=1,N
      DELM(IR,IC)=DELM(IR,IC)-TELM(IR,JN)*A(JN,IC)*DT/(TM+1.0)
  500 WORM(IR,IC)=TERM(IR,JN)*A(JN,IC)*DT/TM+WORM(IR,IC)
      DO 401 IR=1,N
      DO 401 IC=1,N
      TERM(IR,IC)=WORM(IR,IC)
      TELM(IR,IC)=DELM(IR,IC)
      DELP(IR,IC)=DELP(IR,IC)+TELM(IR,IC)
      PHI(IR,IC)=PHI(IR,IC)+TERM(IR,IC)
      DELM(IR,IC)=0.
  401 WORM(IR,IC)=0.0
      ABC=0.0
      DO 2 I=1,N
      DO 2 J=1,N
      AA=TERM(I,J)
      AB=ABSF(AA)
      IF(ABC-AB) 3,3,2
C     FIND BIGGEST VALUE
    3 ABC=AB
    2 CONTINUE
      IF(0.000000005-ABC) 5,5,6
    5 GO TO 4
    6 PRINT 502,((PHI(IR,IC),IC=1,N),IR=1,N)
```

```
  502 FORMAT(///1X,9HPHI(I,J)=/((4F15.9)))
C
      DO 600 I=1,N
      DO 600 K=1,N
      DO 600 J=1,N
  600 DEL(I)=DEL(I)+PHI(I,J)*DELP(J,K)*C(K)
      PRINT 503 (DEL(I),I=1,N)
  503 FORMAT(///1X,7HDEL(I)=//(4F15.9))
      END
      END
```

```
DT=0.10

A(I,J)=
    .00      1.00       .00       .00
  17.20       .00       .00      3.50
    .00       .00       .00      1.00
    .00       .00       .00     -2.00

C(I)=
    .00    -17.20       .00      9.80

PHI(I,J)=
    1.087239756    0.102891421    0.000000000    0.016631936
    1.769732445    1.087239756    0.000000000    0.326856101
    0.000000000    0.000000000     1.00000000     .090634623
    0.000000000    0.000000000     0.00000000     .818730753

DEL(I)=
   -.081750092   -1.606739468    0.045890345    0.888219310
```

```
..JOB0250F,WAN
      PROGRAM OPTCON
C     MINIMUM   J=SUM(XT(K)*Q*X(K)+R*U**2)
      DIMENSION PHI(4,4),PSI(4,4),P(4,4),DEL(4),AT(20,4),
     1GM(4,4),Q(4,4),FM(4),EM(4)
      READ 1,N,NM,NPRINT
    1 FORMAT (8I10)
      READ2,R,DT
      READ2, ((Q(I,J),J=1,N),I=1,N)
      READ2,((PHI(I,J),J=1,N),I=1,N)
      READ2,(DEL(I),I=1,N)
    2 FORMAT((4F16.9))
      PRINT 3,N,NM,NPRINT
    3 FORMAT(///(8I10))
      PRINT 44,R,DT
   44 FORMAT(//1X,2HR=,F5.2,3X,3HDT=,F5.2)
      PRINT 45,((Q(I,J),J=1,N),I=1,N)
   45 FORMAT(//1X,7HQ(I,J)=/((4F16.9)))
      PRINT 46, ((PHI(I,J),J=1,N),I=1,N)
   46 FORMAT(//1X,9HPHI(I,J)=/((4F16.9)))
      PRINT 47, (DEL(I),I=1,N)
   47 FORMAT(//1X,7HDEL(I)=/(4F16.9))
      DO 5I=1,N
      DO5 J=1,N
      GM(I,J)=0.0
      EM(I)=0.0
      FM(I)=0.0
      P(I,J)=Q(I,J)
    5 PSI(I,J)=0.0
C
C     CALCULATE  AT(K,J)
      DO 22 KK=1,NPRINT
      DO 20 K=1,NM
      DEN=0.0
      DO6 I=1,N
      DO 6 J=1,N
    6 EM(I)=EM(I)+DEL(J)*P(J,I)
      DO 8 I=1,N
      DO 7 J=1,N
    7 FM(I)=FM(I)+EM(J)*PHI(J,I)
    8 DEN=DEN+EM(I)*DEL(I)
      DEN=-DEN-R
      DO 10 I=1,N
      AT(K,I)=FM(I)/DEN
      FM(I)=0.0
   10 EM(I)=0.0
C
C     CALCULATE   PSI(K,I,J)
      DO 13 I=1,N
      DO 13 J=1,N
   13 PSI(I,J)=PHI(I,J)+DEL(I)*AT(K,J)
C
C     CALCULATE   P(K,I,J)
      DO 16 I=1,N
```

```
      DO 16 J=1,N
      DO 15 L=1,N
      DO 15 M=1,N
   15 GM(I,J)=GM(I,J)+PSI(L,I)*P(L,M)*PSI(M,J)
      P(I,J)=GM(I,J)+R*AT(K,I)*AT(K,J)
C     FOR TERMINAL CONTROL OMIT Q(I,J)
   16 GM(I,J)=0.0
   20 CONTINUE
C
C     PRINT
      PRINT 12,KK,(AT(NM,J),J=1,N)
   12 FORMAT(//1X,3HAT(,I2,2H)=/(4F16.9))
   22 CONTINUE
      END
      END
```

```
AT(10)=
   2.484604417    .5990829066    .007459609    .344834834
```

```
..JOB0250F,WAN
      PROGRAM BROOM
C     OPTIMAL DISCRETE TIME CONTROL
C     FOR SIMPLICITY, USE AT(J) IN LAST STAGE FOR ALL U
      DIMENSION X(30),XDOT(30),C(15)
      C(10)=1.
    1 CALL INTEG1 (T,X,XDOT,C)
      XDOT(1)=X(2)
      XDOT(2)=17.2*SINF(X(1))+3.5*X(4)-17.2*U
      XDOT(3)=X(4)
      XDOT(4)=-2.*X(4)+9.8*U
      DT=0.1
C     DT = HOLD TIME
      IF(T-C(5)) 1,2,2
C     ABOVE STATEMENT AS A ZERO ORDER HOLD
    2 U=C(1)*X(1)+C(2)*X(2)+C(3)*X(3)+C(4)*X(4)
      X(5)=U
      C(5)=C(5)+DT
      GO TO 1
      END
      END
```